

EXPLICIT UNOBSTRUCTED PRIMES FOR MODULAR DEFORMATION PROBLEMS OF SQUAREFREE LEVEL

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Dedicated to the memory of Arnold Ephraim Ross

1. INTRODUCTION

Let f be a newform of weight $k \geq 2$, level N , and character ω . Let K be the number field generated by the Fourier coefficients of f . For any prime λ of K Deligne has constructed a semisimple Galois representation

$$\bar{\rho}_{f,\lambda} : G_{\mathbf{Q},S \cup \{\ell\}} \rightarrow \mathrm{GL}_2 k_\lambda$$

over the residue field k_λ of K at λ ; here $G_{\mathbf{Q},S \cup \{\ell\}}$ is the Galois group of the maximal extension of \mathbf{Q} unramified outside the set S of places dividing $N\infty$ and the characteristic ℓ of k_λ . The representation $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for almost all primes λ ; we write $\mathrm{Red}(f)$ for the set of λ such that $\bar{\rho}_{f,\lambda}$ is not absolutely irreducible.

Following Mazur, we say that a prime $\lambda \notin \mathrm{Red}(f)$ is an *obstructed prime* for f if the cohomology group $H^2(G_{\mathbf{Q},S \cup \{\ell\}}, \mathrm{ad} \bar{\rho}_{f,\lambda})$ of the adjoint representation of $\bar{\rho}_{f,\lambda}$ is non-zero. We write $\mathrm{Obs}(f)$ for the set of such primes. The importance of this notion rests on the fact that for $\lambda \notin \mathrm{Obs}(f) \cup \mathrm{Red}(f)$, the universal deformation ring associated to $\bar{\rho}_{f,\lambda}$ is isomorphic to a power series ring in three variables over the Witt vectors of k_λ ; see Section 2 for details.

In the general case, it was shown in [21] that $\mathrm{Obs}(f)$ is finite for f of weight $k \geq 3$ and of density zero for f of weight $k = 2$. (Similar results have been obtained in some cases by Yamagami [22].) Previously some explicit results on the set $\mathrm{Obs}(f)$ were obtained in the latter case in [10] and [1]. In this paper we obtain an explicit bound on $\mathrm{Obs}(f)$ in the case that the level N of f is squarefree. We state our result here only for $N > 1$; see Section 4.2 for the general statement (where we also allow $k = 2$ and S non-minimal) and a partial converse.

Theorem 1.1. *Assume that $k \geq 3$ and that $N > 1$ is squarefree. Let M denote the conductor of the Dirichlet character ω . Then*

$$\mathrm{Obs}(f) \subseteq \left\{ \lambda \mid \ell; \ell \leq k + 1 \text{ or } \ell \mid N\varphi(N) \prod_{p \mid \frac{N}{M}} (p + 1) \right\} \cup \mathrm{Cong}(f)$$

with $\mathrm{Cong}(f)$ the set of congruence primes for f (as defined in Section 4.1) and φ the Euler totient function.

We note that the set $\mathrm{Cong}(f)$ is computable using the results of [19] and a tool such as [18]. It is not immediately clear to the author what form to expect the analogue of this result to take for N not squarefree.

In Section 2 we give a brief review of deformation theory and use standard duality arguments to reduce the vanishing of $H^2(G_{\mathbf{Q},S}, \mathrm{ad} \bar{\rho}_{f,\lambda})$ to the vanishing

of certain local and global cohomology groups. The local groups are the subject of Section 3; the computations rest on some simple cases of the local Langlands correspondence. In Section 4.1 we use results of Hida (as refined in [11]) to relate the global cohomology group to a certain Selmer group studied by Diamond, Flach, and Guo. The main results of the paper are proved in Section 4.2. We give several explicit examples in Section 5.

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Notation. If $\rho : G \rightarrow \mathrm{GL}_2 R$ is a representation of a group G over a ring R , we write $\mathrm{ad} \rho : G \rightarrow \mathrm{GL}_4 R$ for the adjoint representation of G on $\mathrm{End}(\rho)$ and $\mathrm{ad}^0 \rho : G \rightarrow \mathrm{GL}_3 R$ for the kernel of the trace map from $\mathrm{ad} \rho$ to the trivial representation. If $\rho : G \rightarrow \mathrm{GL}_n R$ is any representation, we write $H^i(G, \rho)$ for the cohomology group $H^i(G, V_\rho)$ with V_ρ a free R -module of rank n with G -action via ρ . We write ρ^{ss} for the semisimplification of such a ρ .

We write $G_{\mathbf{Q}}$ for the absolute Galois group of \mathbf{Q} . We fix now and forever embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ for each p , yielding injections $G_p \hookrightarrow G_{\mathbf{Q}}$ with G_p the absolute Galois group of $\bar{\mathbf{Q}}_p$. We write I_p for the inertia subgroup of G_p . Let $\varepsilon_\ell : G_{\mathbf{Q}} \rightarrow \mathbf{Z}_\ell^\times$ be the ℓ -adic cyclotomic character and let $\bar{\varepsilon}_\ell : G_{\mathbf{Q}} \rightarrow \mathbf{F}_\ell^\times$ be its reduction, the mod ℓ Teichmüller character. If M is a $\mathbf{Z}_\ell[G_{\mathbf{Q}}]$ -module, we write $M(1)$ for its first Tate twist $M \otimes_{\mathbf{Z}_\ell} \varepsilon_\ell$. If S is a set of places of \mathbf{Q} containing the infinite place, the expression “ $p \in S$ ” is to be interpreted as “ $p \in S - \{\infty\}$ ”.

2. OBSTRUCTIONS

2.1. Deformation theory. In this section we review the fundamentals of the deformation theory of representations of profinite groups as introduced in [14] and generalized in [3]; see also [7] for a quick proof of representability in the most general case. Let k be a finite field and let \mathcal{C} denote the category of local rings which are inverse limits of artinian local rings with residue field k ; a morphism $A \rightarrow B$ in \mathcal{C} is a continuous local homomorphism inducing the identity map on residue fields. Note that any ring A in \mathcal{C} is canonically an algebra for the Witt vectors $W(k)$ of k .

Let G be a profinite group and fix an absolutely irreducible continuous representation

$$\bar{\rho} : G \rightarrow \mathrm{GL}_n k$$

for some $n \geq 1$. A *lifting* of $\bar{\rho}$ to a ring A in \mathcal{C} is a continuous representation $\rho : G \rightarrow \mathrm{GL}_n A$ such that the composition

$$G \xrightarrow{\rho} \mathrm{GL}_n A \longrightarrow \mathrm{GL}_n k$$

is equal to $\bar{\rho}$. Two liftings ρ_1, ρ_2 of $\bar{\rho}$ are said to be *strictly equivalent* if there is a matrix M in the kernel of $\mathrm{GL}_n A \rightarrow \mathrm{GL}_n k$ such that $\rho_1 = M \cdot \rho_2 \cdot M^{-1}$.

A *deformation* of $\bar{\rho}$ to A is a strict equivalence class of liftings. Let

$$D_{\bar{\rho}} : \mathcal{C} \rightarrow \mathrm{Sets}$$

be the functor sending a ring A to the set of deformations of $\bar{\rho}$ to A . The deformation functor $D_{\bar{\rho}}$ is representable by [3, Theorem 2.3]; that is, there is a ring $R_{\bar{\rho}}$ in \mathcal{C} (called the *universal deformation ring* of $\bar{\rho}$) and an isomorphism of functors

$$(2.1) \quad D_{\bar{\rho}}(-) \cong \mathrm{Hom}_{\mathcal{C}}(R_{\bar{\rho}}, -).$$

Note that via (2.1) the identity map on $R_{\bar{\rho}}$ corresponds to a deformation

$$\rho^{\text{univ}} : G \rightarrow \text{GL}_n R_{\bar{\rho}}$$

of $\bar{\rho}$ to $R_{\bar{\rho}}$; this is the *universal deformation* of $\bar{\rho}$, and the isomorphism (2.1) sends $f : R_{\bar{\rho}} \rightarrow A$ to the deformation $f \circ \rho^{\text{univ}}$ of $\bar{\rho}$ to A .

The next proposition gives the fundamental connection between the deformation problem $D_{\bar{\rho}}$ and the cohomology groups $H^i(G, \text{ad } \bar{\rho})$. We say that $D_{\bar{\rho}}$ is *unobstructed* if $H^2(G, \text{ad } \bar{\rho}) = 0$.

Proposition 2.1. *Assume that $H^i(G, \text{ad } \bar{\rho})$ is finite-dimensional over k for each i ; set $d = \dim_k H^1(G, \text{ad } \bar{\rho})$. Then there exists a (non-canonical) surjection*

$$(2.2) \quad W(k)[[T_1, \dots, T_d]] \twoheadrightarrow R_{\bar{\rho}}$$

with kernel generated by at most $\dim_k H^2(G, \text{ad } \bar{\rho})$ elements. In particular, if $D_{\bar{\rho}}$ is unobstructed, then (2.2) is an isomorphism.

Proof. This is proved in [14, Section 1.6]. The existence of the surjection (2.2) follows from an isomorphism

$$D_{\bar{\rho}}(k[\epsilon]/\epsilon^2) \xrightarrow{\cong} H^1(G, \text{ad } \bar{\rho})$$

(sending a deformation ρ to the cocycle c_ρ such that $\rho(g) = \bar{\rho}(g)(1 + \epsilon \cdot c_\rho(g))$ for all $g \in G$) and the interpretation of these groups as the tangent space of $R_{\bar{\rho}}$ via (2.1). The statement about the kernel J of (2.2) follows from an injection

$$\text{Hom}(J, k) \hookrightarrow H^2(G, \text{ad } \bar{\rho})$$

constructed using an obstruction two-cocycle measuring the failure of ρ^{univ} to lift via (2.2). □ □

The next lemma will be useful later in the paper.

Lemma 2.2. *Let $\bar{\rho} : G \rightarrow \text{GL}_2 k$ be continuous and absolutely irreducible and let $\chi : G \rightarrow k^\times$ be a character of order at least 3. Then $H^0(G, \chi \otimes \text{ad } \bar{\rho}) = 0$.*

Proof. If the projective image of $\bar{\rho}$ is dihedral, then the G -representation $\text{ad } \bar{\rho}$ is the sum of the trivial character, a quadratic character, and an irreducible two-dimensional representation of G . If the projective image of $\bar{\rho}$ is not dihedral, then $\text{ad } \bar{\rho}$ is the sum of the trivial character and an irreducible three-dimensional representation of G . In either case the lemma follows since χ is neither trivial nor quadratic. □ □

2.2. Galois cohomology. Let k be a finite field of odd characteristic ℓ . We now apply the discussion of the previous section to the case of a two-dimensional Galois representation over k . Fix a finite set S of places of \mathbf{Q} including ℓ and the infinite place. Let \mathbf{Q}_S denote the maximal extension of \mathbf{Q} unramified outside S ; set $G_{\mathbf{Q}, S} := \text{Gal}(\mathbf{Q}_S/\mathbf{Q})$. Let

$$\bar{\rho} : G_{\mathbf{Q}, S} \rightarrow \text{GL}_2 k$$

be continuous and absolutely irreducible. We assume further that $\bar{\rho}$ is *odd* in the sense that the image of complex conjugation has distinct eigenvalues. In this section we study the cohomology groups $H^i(G_{\mathbf{Q}, S}, \text{ad } \bar{\rho})$.

Lemma 2.3. *Each cohomology group $H^i(G_{\mathbf{Q}, S}, \text{ad } \bar{\rho})$ is finite-dimensional over k and*

$$\dim_k H^1(G_{\mathbf{Q}, S}, \text{ad } \bar{\rho}) - \dim_k H^2(G_{\mathbf{Q}, S}, \text{ad } \bar{\rho}) = 3.$$

Proof. The first statement is [15, Corollary 4.15], while the second is a straightforward calculation using Tate's global Euler characteristic formula as in [14, Section 1.10]. \square \square

Corollary 2.4. *If $H^2(G_{\mathbf{Q},S}, \text{ad } \bar{\rho}) = 0$, then the universal deformation ring $R_{\bar{\rho}}$ is (non-canonically) isomorphic to $W(k)[[T_1, T_2, T_3]]$.*

It remains to study $H^2(G_{\mathbf{Q},S}, \text{ad } \bar{\rho})$; we will do this using global duality theorems of Poitou and Tate. For a $k[G_{\mathbf{Q},S}]$ -module M , define

$$\text{III}^1(G_{\mathbf{Q},S}, M) := \ker(H^1(G_{\mathbf{Q},S}, M) \rightarrow \bigoplus_{p \in S} H^1(G_p, M)).$$

Lemma 2.5. *One has*

$$\dim_k H^2(G_{\mathbf{Q},S}, \text{ad } \bar{\rho}) \leq \dim_k \text{III}^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho}) + \sum_{p \in S} \dim_k H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho})$$

with equality if $\ell \neq 3$.

Proof. The trace pairing $\text{ad } \bar{\rho} \otimes \text{ad } \bar{\rho} \rightarrow k$ identifies $\bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}$ with the Cartier dual of $\text{ad } \bar{\rho}$. Thus by [15, Theorem 4.10] there is an exact sequence

$$0 \rightarrow H^0(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}) \rightarrow \bigoplus_{p \in S} H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}) \rightarrow \text{Hom}(H^2(G_{\mathbf{Q},S}, \text{ad } \bar{\rho}), k) \rightarrow \text{III}^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}) \rightarrow 0.$$

Since $\bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho} = \bar{\varepsilon}_\ell \oplus (\bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho})$ and $\text{III}^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell)$ vanishes by [20, Lemma 10.6], the lemma follows from the exact sequence and Lemma 2.2. \square \square

We will study the local terms $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho})$ in Section 3. The global term $\text{III}^1(G_{\mathbf{Q},S}, \bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho})$ is difficult to control directly; instead we now relate it to a certain Selmer group, which in turn is often computable using the results of Section 4.1.

Fix a totally ramified extension K of the field of fractions of $W(k)$. The ring of integers \mathcal{O} of K lies in \mathcal{C} ; we write \mathfrak{m} for its maximal ideal. Let $\rho : G_{\mathbf{Q},S} \rightarrow \text{GL}_2 \mathcal{O}$ be a lifting of $\bar{\rho}$ to \mathcal{O} . Let V_ρ (resp. A_ρ) denote a three-dimensional K -vector space (resp. $(K/\mathcal{O})^3$) endowed with a $G_{\mathbf{Q},S}$ -action via $\text{ad}^0 \rho : G_{\mathbf{Q},S} \rightarrow \text{GL}_3 \mathcal{O}$.

Let V (resp. A) denote either V_ρ (resp. A_ρ) or else its Tate twist. For a prime p , define

$$H_f^1(G_p, V) := \begin{cases} H^1(G_p/I_p, V^{I_p}) & p \neq \ell; \\ \ker(H^1(G_p, V) \rightarrow H^1(G_p, V \otimes B_{\text{crys}})) & p = \ell; \end{cases}$$

regarded as a K -subspace of $H^1(G_p, V)$; here B_{crys} is the crystalline period ring of Fontaine. Let $H_f^1(G_p, A)$ denote the image of $H_f^1(G_p, V)$ under the pushforward from $H^1(G_p, V)$ to $H^1(G_p, A)$. For M denoting either of V or A , the *Selmer group* of M is defined by

$$H_f^1(G_{\mathbf{Q}}, M) := \{c \in H^1(G_{\mathbf{Q}}, M); c|_{G_p} \in H_f^1(G_p, M) \text{ for all } p\}.$$

Following [5, Section 7], we will also need a slight variant of this construction. Define

$$H_w^1(G_p, A) := \begin{cases} H^1(G_p/I_p, A^{I_p}) & p \neq \ell; \\ H_f^1(G_\ell, A) & p = \ell; \end{cases}$$

$$H_\emptyset^1(G_{\mathbf{Q}}, A) := \{c \in H^1(G_{\mathbf{Q}}, A); c|_{G_p} \in H_w^1(G_p, A) \text{ for all } p\}.$$

Clearly one has $H_f^1(G_p, A) \subseteq H_w^1(G_p, A)$ for all p , so that

$$(2.3) \quad H_f^1(G_{\mathbf{Q}}, A) \subseteq H_{\theta}^1(G_{\mathbf{Q}}, A).$$

In fact, this inclusion is an equality if A^{I_p} is divisible for all $p \neq \ell$.

Lemma 2.6. *Assume that $\ell > 3$ and $H_f^1(G_{\mathbf{Q}}, V_{\rho}) = H_f^1(G_{\mathbf{Q}}, V_{\rho}(1)) = 0$. Then*

$$\dim_k \text{III}^1(G_{\mathbf{Q}, S}, \bar{\varepsilon}_{\ell} \otimes \text{ad}^0 \bar{\rho}) \leq \dim_k H_{\theta}^1(G_{\mathbf{Q}, S}, A_{\rho})[\mathfrak{m}].$$

Proof. Since ρ is a lifting of $\bar{\rho}$, the $k[G_{\mathbf{Q}, S}]$ -module $A_{\rho}(1)[\mathfrak{m}]$ is a realization of $\bar{\varepsilon}_{\ell} \otimes \text{ad}^0 \bar{\rho}$. We thus obtain a natural map

$$(2.4) \quad H^1(G_{\mathbf{Q}}, \bar{\varepsilon}_{\ell} \otimes \text{ad}^0 \bar{\rho}) = H^1(G_{\mathbf{Q}}, A_{\rho}(1)[\mathfrak{m}]) \rightarrow H^1(G_{\mathbf{Q}}, A_{\rho}(1))$$

which is injective by Lemma 2.2. The image of $\text{III}^1(G_{\mathbf{Q}, S}, \bar{\varepsilon}_{\ell} \otimes \text{ad}^0 \bar{\rho})$ under (2.4) is easily seen to lie in $H_f^1(G_{\mathbf{Q}}, A_{\rho}(1))$, so that we obtain an injection

$$(2.5) \quad \text{III}^1(G_{\mathbf{Q}, S}, \bar{\varepsilon}_{\ell} \otimes \text{ad}^0 \bar{\rho}) \hookrightarrow H_f^1(G_{\mathbf{Q}}, A_{\rho}(1)).$$

By [9, Theorem 1], the latter group is dual to $H_f^1(G_{\mathbf{Q}}, A_{\rho})$ (see [21, Proposition 2.2]; this also uses the assumption on the vanishing of the Selmer group of V_{ρ} and $V_{\rho}(1)$). The lemma thus follows from (2.5) and (2.3). \square \square

Remark 2.7. The only difficulty in analyzing the failure of (2.5) to be an isomorphism on \mathfrak{m} -torsion is the determination of the image of the restriction map

$$H_f^1(G_{\mathbf{Q}}, A_{\rho}(1))[\mathfrak{m}] \rightarrow H_f^1(G_{\ell}, A_{\rho}(1))[\mathfrak{m}].$$

Unfortunately, this question appears to be quite difficult in general.

3. LOCAL INVARIANTS

Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, squarefree level N , and character ω . Let K be the number field generated by the Fourier coefficients a_n of f ; note that by [16, Corollary 3.1] it also contains the image of ω . For any prime λ of K , Deligne has constructed a continuous λ -adic Galois representation

$$\rho_{f, \lambda} : G_{\mathbf{Q}} \rightarrow \text{GL}_2 K_{\lambda}.$$

This representation is unramified at $p \nmid N\ell$ (with ℓ the characteristic of the residue field k_{λ} of K_{λ}) and for such p the trace (resp. the determinant) of the image of an arithmetic Frobenius element Frob_p under $\rho_{f, \lambda}$ is equal to a_p (resp. $p^{k-1}\omega(p)$).

As usual we identify $\omega : (\mathbf{Z}/N\mathbf{Z})^{\times} \rightarrow \mu_{\varphi(N)}$ with a Galois character via the canonical isomorphism $\text{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q}) \cong (\mathbf{Z}/N\mathbf{Z})^{\times}$; the determinant of $\rho_{f, \lambda}$ is then $\varepsilon_{\ell}^{k-1}\omega$. Let M denote the conductor of ω and let $\omega_0 : (\mathbf{Z}/M\mathbf{Z})^{\times} \rightarrow \mu_{\varphi(M)}$ be the associated primitive Dirichlet character. Then ω is ramified at p if and only if p divides M , in which case the restriction of ω to the inertia group I_p is a non-trivial character taking values in μ_{p-1} .

For the remainder of this section we fix a prime λ of K dividing a rational prime ℓ . Let

$$\bar{\rho}_{f, \lambda} : G_{\mathbf{Q}} \rightarrow \text{GL}_2 k_{\lambda}$$

be the semisimple reduction of $\rho_{f, \lambda}$; this is well-defined (up to conjugation) independent of any choice of integral model of $\rho_{f, \lambda}$. We are interested in the local invariants $H^0(G_p, \bar{\varepsilon}_{\ell} \otimes \text{ad} \bar{\rho}_{f, \lambda})$ for all primes p . As

$$\bar{\varepsilon}_{\ell} \otimes \text{ad} \bar{\rho}_{f, \lambda} \cong \bar{\varepsilon}_{\ell} \oplus (\bar{\varepsilon}_{\ell} \otimes \text{ad}^0 \bar{\rho}_{f, \lambda})$$

and

$$H^0(G_p, \bar{\varepsilon}_\ell) \neq 0 \Leftrightarrow p \equiv 1 \pmod{\ell},$$

we will restrict our attention below to the case that ℓ does not divide $p - 1$ and to the study of $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho}_{f,\lambda})$.

In the analysis below we make use of the local Langlands correspondence and the compatibility results completed in [2], together with the analysis of possible reduction types in [6, Section 1]. Rather than review these results in detail, we will only recall the consequences we need; see [21] for more details and references.

3.1. $p \nmid N\ell$. Fix $\alpha_p, \beta_p \in \bar{K}$ with $\alpha_p + \beta_p = a_p$ and $\alpha_p \beta_p = p^{k-1} \omega(p)$. In this case, we have

$$\rho_{f,\lambda}|_{G_p}^{\text{ss}} \otimes \bar{K}_\lambda \cong \chi_1 \oplus \chi_2$$

where the $\chi_i : G_p \rightarrow \bar{K}_\lambda^\times$ are unramified characters with

$$(3.1) \quad \chi_1(\text{Frob}_p) = \alpha_p; \quad \chi_2(\text{Frob}_p) = \beta_p.$$

We write $\bar{\chi}_i : G_p \rightarrow \bar{k}_\lambda^\times$ for the reduction of χ_i . In fact, $\rho_{f,\lambda}|_{G_p} \otimes \bar{K}_\lambda$ is split so long as $\alpha_p \neq \beta_p$; if also $\alpha_p \not\equiv \beta_p \pmod{\lambda}$, then necessarily

$$\bar{\rho}_{f,\lambda}|_{G_p} \otimes \bar{k}_\lambda \cong \bar{\chi}_1 \oplus \bar{\chi}_2.$$

(See [6, Section 1] for details.)

Lemma 3.1. *Assume $p \nmid N\ell$ and $p \not\equiv 1 \pmod{\ell}$. Then $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad} \bar{\rho}_{f,\lambda}) \neq 0$ if and only if $a_p^2 \equiv (p+1)^2 p^{k-2} \omega(p) \pmod{\lambda}$.*

Proof. Since the existence of eigenvectors with k_λ -rational eigenvalues is invariant under base extension, to show that $\bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}$ has no G_p -invariants it suffices to show that

$$(3.2) \quad (\bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}|_{G_p}^{\text{ss}}) \otimes \bar{k}_\lambda \cong \bar{\varepsilon}_\ell \oplus \bar{\varepsilon}_\ell \bar{\chi}_1 \bar{\chi}_2^{-1} \oplus \bar{\varepsilon}_\ell \bar{\chi}_1^{-1} \bar{\chi}_2$$

has no G_p -invariants. As $p \not\equiv 1 \pmod{\ell}$, (3.2) has non-trivial G_p -invariants if and only if one of the characters $\bar{\varepsilon}_\ell \bar{\chi}_1 \bar{\chi}_2^{-1}$, $\bar{\varepsilon}_\ell \bar{\chi}_1^{-1} \bar{\chi}_2$ is trivial. By (3.1) this occurs if and only if

$$(3.3) \quad \frac{\alpha_p}{\beta_p} \equiv p^{\pm 1} \pmod{\lambda}.$$

This in turn is equivalent to

$$(3.4) \quad \begin{aligned} \frac{\alpha_p}{\beta_p} + \frac{\beta_p}{\alpha_p} &\equiv p + \frac{1}{p} \pmod{\lambda} \\ \frac{(\alpha_p + \beta_p)^2}{\alpha_p \beta_p} &\equiv \frac{(p+1)^2}{p} \pmod{\lambda} \\ a_p^2 &\equiv (p+1)^2 p^{k-2} \omega(p) \pmod{\lambda} \end{aligned}$$

as claimed.

For the converse, note that if (3.4) holds, then (3.3) and our assumption that $p \not\equiv 1 \pmod{\ell}$ force $\alpha_p \not\equiv \beta_p \pmod{\lambda}$. Thus (3.2) already holds without semi-simplification; the converse now follows easily. \square \square

3.2. $p \mid M$, $p \neq \ell$. In this case the p -component π_p of the automorphic representation associated to f has conductor 1 and ramified central character. It follows that π_p is a principal series representation associated to one ramified character and one unramified character. On the Galois side, this translates to

$$\rho_{f,\lambda}|_{G_p}^{\text{ss}} \otimes \bar{K}_\lambda \cong \chi_1 \oplus \chi_2$$

for continuous characters $\chi_i : G_p \rightarrow \bar{K}_\lambda^\times$ with χ_1 ramified and χ_2 unramified. Since $\rho_{f,\lambda}|_{G_p}$ has determinant $\varepsilon_\ell^{k-1}\omega|_{G_p}$, we have $\chi_1\chi_2 = \varepsilon_\ell^{k-1}\omega|_{G_p}$. In particular $\chi_1|_{I_p} = \omega|_{I_p}$ is a non-trivial character taking values in μ_{p-1} . If $p \not\equiv 1 \pmod{\ell}$, then μ_{p-1} injects into k_λ^\times and consequently the reduction $\bar{\chi}_1 : G_p \rightarrow \bar{k}_\lambda^\times$ is still ramified at p .

Lemma 3.2. *Assume $p \mid M$, $p \neq \ell$, and $p \not\equiv 1 \pmod{\ell}$. Then $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) = 0$.*

Proof. As in Lemma 3.1, it suffices to show that the two characters $\bar{\varepsilon}_\ell \bar{\chi}_1 \bar{\chi}_2^{-1}$, $\bar{\varepsilon}_\ell \bar{\chi}_1^{-1} \bar{\chi}_2$ are non-trivial. Since $\bar{\varepsilon}_\ell$ and $\bar{\chi}_2$ are unramified at p while $\bar{\chi}_1$ is ramified at p , this is clear. \square \square

3.3. $p \mid \frac{N}{M}$, $p \neq \ell$. In this case π_p has conductor 1 and unramified central character. It follows that π_p is the special representation associated to an unramified character. This means that there exists an unramified character $\chi : G_p \rightarrow \bar{K}_\lambda^\times$ such that

$$(3.5) \quad \rho_{f,\lambda}|_{G_p} \otimes \bar{K}_\lambda \cong \begin{pmatrix} \varepsilon_\ell \chi & * \\ 0 & \chi \end{pmatrix}$$

with the upper right corner ramified.

Lemma 3.3. *Assume $p \mid \frac{N}{M}$, $p \neq \ell$, and $p^2 \not\equiv 1 \pmod{\ell}$. Then $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) \neq 0$ if and only if $\bar{\rho}_{f,\lambda}$ is unramified at p .*

Proof. Since $p^2 \not\equiv 1 \pmod{\ell}$, by [21, Lemma 5.1] we have

$$\bar{\rho}_{f,\lambda}|_{G_p} \otimes \bar{k}_\lambda \cong \begin{pmatrix} \bar{\varepsilon}_\ell \bar{\chi} & \nu \\ 0 & \bar{\chi} \end{pmatrix}$$

for some $\nu : G_p \rightarrow \bar{k}_\lambda$; in fact, one checks directly that $\bar{\chi}^{-1}\nu$ is naturally an element of $H^1(G_p, \bar{k}_\lambda(1))$. Since $\bar{\varepsilon}_\ell$ and $\bar{\chi}$ are unramified, $\bar{\rho}_{f,\lambda}|_{G_p}$ is unramified if and only if $\bar{\chi}^{-1}\nu$ is unramified. However, since $p \not\equiv 1 \pmod{\ell}$ every non-zero element of $H^1(G_p, \bar{k}_\lambda(1))$ is ramified. We conclude that $\bar{\rho}_{f,\lambda}|_{G_p}$ is unramified if and only if it is semisimple. [21, Lemma 5.2] now completes the proof. \square \square

Lemma 3.4. *Assume $p \mid \frac{N}{M}$, $p \neq \ell$, $p^2 \not\equiv 1 \pmod{\ell}$, and $\bar{\rho}_{f,\lambda}$ absolutely irreducible. Then $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) \neq 0$ if and only if there exists a newform f' , of weight k and level dividing $\frac{N}{p}$, such that $\bar{\rho}_{f',\bar{\lambda}} \cong \bar{\rho}_{f,\bar{\lambda}}$ for some prime $\bar{\lambda}$ of $\bar{\mathbf{Q}}$ over λ .*

Here by $\bar{\rho}_{f',\bar{\lambda}}$ (resp. $\bar{\rho}_{f,\bar{\lambda}}$) we mean $\bar{\rho}_{f',\lambda} \otimes \bar{k}_\lambda$ (resp. $\bar{\rho}_{f,\lambda} \otimes \bar{k}_\lambda$), with λ' the intersection of $\bar{\lambda}$ with the field K' of Fourier coefficients of f' and with k'_λ , the residue field of K' at λ' .

Proof. By [8, (B) of p. 221], the existence of such an f' is equivalent to $\bar{\rho}_{f,\lambda}$ being unramified at p . Thus the lemma follows from Lemma 3.3. \square \square

Remark 3.5. If one further assumes that $p' \not\equiv 1 \pmod{\ell}$ for all p' dividing N , then the newform f' of Lemma 3.4 must have level a multiple of M and character lifting ω_0 , so that λ is a congruence prime for f of level dividing $\frac{N}{p}$ in the terminology of Section 4.1. Indeed, $\bar{\rho}_{f',\lambda}$ is isomorphic to $\bar{\rho}_{f,\lambda}$ and thus has determinant $\bar{\varepsilon}_\ell^{k-1}\bar{\omega}$; therefore the character ω' of f' must have reduction equal to $\bar{\omega}$. However, since $p \not\equiv 1 \pmod{\ell}$ for all p dividing N , the only such characters of conductor dividing N are those which lift ω_0 . Thus f' must have level divisible by M and character lifting ω_0 , as claimed.

3.4. $p = \ell$, $\ell \nmid N$. We now give some mild improvements on the results of [21, Section 4] on the vanishing of $H^0(G_\ell, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda})$. Recall that $f = \sum a_n q^n$ is said to be *ordinary* (resp. *supersingular*) at λ if $v_\lambda(a_\ell) = 0$ (resp. $v_\lambda(a_\ell) > 0$), with v_λ the λ -adic valuation. If f is ordinary at λ , then the semisimplification of $\rho_{f,\lambda}|_{I_\ell} \otimes \bar{K}_\lambda$ is isomorphic to $\bar{\varepsilon}_\ell^{k-1} \oplus 1$, while if f is supersingular at λ , then $\bar{\rho}_{f,\lambda}|_{G_\ell}$ is absolutely irreducible. (This all follows from the discussion of [8, pp. 214–215], for example.)

Lemma 3.6. *Assume $\ell \nmid N$. If f is ordinary at λ and $H^0(G_\ell, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) \neq 0$, then $k \equiv 0, 2 \pmod{\ell - 1}$.*

Proof. It suffices to prove the corresponding result for the I_ℓ -invariants of the semisimplification of $(\bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) \otimes \bar{k}_\lambda$. By the above discussion this semisimplification is isomorphic to

$$\bar{\varepsilon}_\ell \oplus \bar{\varepsilon}_\ell \oplus \bar{\varepsilon}_\ell^k \oplus \bar{\varepsilon}_\ell^{2-k}.$$

Since $\bar{\varepsilon}_\ell$ has order $\ell - 1$, the lemma follows. \square \square

Note that the above lemma is vacuous in the case of weight 2.

Lemma 3.7. *Assume $\ell \nmid N$. If f is supersingular at λ and $\ell > 3$, then $H^0(G_\ell, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) = 0$.*

Proof. By the above discussion, $\bar{\rho}_{f,\lambda}|_{G_\ell}$ is absolutely irreducible; thus the lemma is immediate from Lemma 2.2. \square \square

4. GLOBAL RESULTS

We continue with a newform $f = \sum a_n q^n$ of weight k , squarefree level N , and character ω of conductor M as in Section 3. Let \mathcal{O} be the ring of integers of the field K of Fourier coefficients of f . For each prime λ of K , let $V_{\rho,\lambda}$ be a three-dimensional K_λ -vector space with $G_{\mathbf{Q}}$ -action via $\text{ad}^0 \rho_{f,\lambda}$. Fix a $G_{\mathbf{Q}}$ -stable \mathcal{O}_λ -lattice $T_{\rho,\lambda}$ in $V_{\rho,\lambda}$ and set $A_{\rho,\lambda} := V_{\rho,\lambda}/T_{\rho,\lambda}$. In general the $k_\lambda[G_{\mathbf{Q}}]$ -module $A_{\rho,\lambda}[\lambda]$ need not agree with the semisimple reduction $\bar{\rho}_{f,\lambda}$; however, these two representations must be isomorphic when $\bar{\rho}_{f,\lambda}$ is absolutely irreducible, which is the only case we will consider below.

4.1. Congruences and Selmer groups. The purpose of this section is to explain how the results of [12] (as refined in [11]) and [5] relate adjoint Selmer groups with congruences of modular forms. Let d be a divisor of N which is divisible by M . We say that a prime λ of K is a *congruence prime of level d* for f if there exists a newform f' of weight k and level d such that:

- (1) f' has character lifting ω_0 ;
- (2) f' is not a Galois conjugate of f ;
- (3) $\bar{\rho}_{f,\lambda} \cong \bar{\rho}_{f',\lambda}$ for some prime λ of $\bar{\mathbf{Q}}$ above λ .

(Of course, by the Chebotarev density theorem the last condition is equivalent to a congruence $a_p(f) \equiv a_p(f') \pmod{\bar{\lambda}}$ for all primes p not dividing N .)

We say that a congruence prime λ of level d for f is *proper* (resp. *strict*) if $d < N$ (resp. $d = N$). Let $\text{Cong}(f)$ (resp. $\text{Cong}_{<N}(f)$, resp. $\text{Cong}_N(f)$) denote the set of congruence primes (resp. proper congruence primes, resp. strict congruence primes) for f .

We need a simple lemma.

Lemma 4.1. *Let λ be a prime of K dividing a rational prime ℓ not dividing N . Assume that $N > 1$ and that $\bar{\rho}_{f,\lambda}$ is ramified at some p dividing N . If $\bar{\rho}_{f,\lambda}$ is absolutely irreducible, then $\bar{\rho}_{f,\lambda}|_{G_F}$ is absolutely irreducible as well, where $F = \mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$.*

Proof. As in [5, Lemma 7.14], if $\bar{\rho}_{f,\lambda}|_{G_F}$ is absolutely reducible, then $\bar{\rho}_{f,\lambda}$ is induced from a character of G_F . In particular, it follows that the conductor N' of $\bar{\rho}_{f,\lambda}$ (in the sense of [8]) is a square. However, N' must also divide the level N of f ; since N' is non-trivial by hypothesis and N is squarefree, this is impossible. \square \square

Proposition 4.2. *Let λ be a prime of K dividing the rational prime ℓ . Assume that:*

- (1) $\bar{\rho}_{f,\lambda}$ is absolutely irreducible;
- (2) $\ell > k$;
- (3) Either $N > 1$ or $\ell \nmid (2k-3)(2k-1)$;
- (4) $\ell \nmid N$;
- (5) $\ell \nmid \varphi(N)$ (that is, $p \not\equiv 1 \pmod{\ell}$ for all $p \mid N$);
- (6) $\bar{\rho}_{f,\lambda}$ is ramified at p for all $p \mid \frac{N}{M}$;

Then $H_\theta^1(G_{\mathbf{Q}}, A_{\rho,\lambda}) \neq 0$ if and only if $\lambda \in \text{Cong}_N(f)$.

Proof. Conditions (4)–(6) guarantee that $A_{\rho,\lambda}$ is minimally ramified in the sense of [4, Section 3]. Using (1), (2), (4), and Lemma 4.1 (or (3) and [5, Lemma 7.14] for $N = 1$), we may apply [5, Theorem 7.15] to conclude that

$$(4.1) \quad \text{length}_{\mathcal{O}_\lambda} H_\theta^1(G_{\mathbf{Q}}, A_{\rho,\lambda}) = v_\lambda(\eta_f^\theta);$$

here η_f^θ is the fractional ideal of K defined in [5, Section 6.4] and v_λ is the λ -adic valuation.

By definition, the ideal η_f^θ is generated by the discriminant $d(L_f(\mathcal{O}_K))$ of [11, proof of Theorem 5], which in turn equals the square of the algebraic special value of the adjoint L -function of f :

$$(4.2) \quad d(L_f(\mathcal{O}_K)) = \left(\frac{W(f)\Gamma(1, \text{ad } f)L(1, \text{ad } f)}{\Omega(f, +)\Omega(f, -)} \right)^2.$$

(All of this is only true up to factors of primes violating (1)–(4).) In particular, (4.1) implies that $H_\theta^1(G_{\mathbf{Q}}, A_{\rho,\lambda})$ is non-zero if and only if (4.2) has positive λ -adic valuation. By [11, Theorems 1 and 2], the latter condition is equivalent to the existence of a newform f' of weight k and level dividing N , not Galois conjugate to f , such that $\bar{\rho}_{f,\bar{\lambda}} \cong \bar{\rho}_{f',\bar{\lambda}}$ for some prime $\bar{\lambda}$ above λ .

It remains to show that f' has level N and character ω . Since $\bar{\rho}_{f',\bar{\lambda}}$ has determinant $\varepsilon_\ell^{k-1}\bar{\omega}$ and $\mu_{\varphi(N)}$ injects into k_λ^\times (by (5)), f' has level divisible by M and character lifting ω_0 . Hypothesis (6) guarantees that $\bar{\rho}_{f',\bar{\lambda}}$ is ramified at all $p \mid \frac{N}{M}$ as well, so that f' must in fact have level N . \square \square

4.2. Vanishing of cohomology. Let S be a finite set of places of \mathbf{Q} containing all places dividing $N\infty$; let N_S denote the product of all primes in S . Fix a prime λ of K dividing a rational prime ℓ . We are now in a position to compute $H^2(G_{\mathbf{Q}, S \cup \{\ell\}}, \text{ad } \bar{\rho}_{f, \lambda})$.

Theorem 4.3. *Assume that $\bar{\rho}_{f, \lambda}$ is absolutely irreducible and $\ell > 3$. If*

$$(4.3) \quad H^2(G_{\mathbf{Q}, S \cup \{\ell\}}, \text{ad } \bar{\rho}_{f, \lambda}) \neq 0,$$

then one of the following holds:

- (1) $\ell \leq k$;
- (2) $\ell \mid N$;
- (3) $\ell \mid \varphi(N_S)$;
- (4) $\ell \mid p + 1$ for some $p \mid \frac{N}{M}$;
- (5) $a_p^2 \equiv (p + 1)^2 p^{k-2} \omega(p) \pmod{\lambda}$ for some $p \mid \frac{N_S}{N}$, $p \neq \ell$;
- (6) $\ell = k + 1$ and f is ordinary at λ ;
- (7) $k = 2$ and $a_\ell^2 \equiv \omega(\ell) \pmod{\lambda}$;
- (8) $N = 1$ and $\ell \mid (2k - 3)(2k - 1)$;
- (9) $\lambda \in \text{Cong}(f)$.

Using Lemma 2.5 and the results of Sections 3 and 4.1, the reader should have little difficulty in detecting the source of each of the conditions above. We shall nevertheless attempt to give a complete proof.

Proof. If (4.3) holds, then Lemma 2.5 implies that either

$$(4.4) \quad H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f, \lambda}) \neq 0$$

for some $p \in S \cup \{\ell\}$ or

$$(4.5) \quad \text{III}^1(G_{\mathbf{Q}, S}, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f, \lambda}) \neq 0.$$

Suppose first that (4.4) holds for a prime $p \in S \cup \{\ell\}$; we may assume $\ell \nmid N$ by (2). If $H^0(G_p, \bar{\varepsilon}_\ell) \neq 0$, then ℓ divides $p - 1$ which in turn divides $\varphi(N_S)$, so that (3) holds. We may thus assume that $p \not\equiv 1 \pmod{\ell}$ and

$$H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho}_{f, \lambda}) \neq 0.$$

By Lemma 3.2 we know that p does not divide M . If p divides $\frac{N}{M}$, then by Lemma 3.4 and Remark 3.5 one of (4) or (9) holds, while if p does not divide $N\ell$, then Lemma 3.1 implies that (5) must hold. Finally, if $p = \ell$ and $k > 2$, then Lemmas 3.6 and 3.7 force (1) or (6) to hold; if $k = 2$, then [21, Proposition 4.4] forces (7) to hold.

It remains to consider the case that (4.5) holds, (4.4) does not hold for any $p \in S \cup \{\ell\}$, and none of (1)–(8) hold. Then by [5, Theorem 8.2]

$$H_f^1(G_{\mathbf{Q}}, V_{f, \lambda}) = H_f^1(G_{\mathbf{Q}}, V_{f, \lambda}(1)) = 0.$$

Lemma 2.6 and (4.5) thus imply that

$$H_\emptyset^1(G_{\mathbf{Q}}, A_{\rho, \lambda}) \neq 0.$$

Since $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f, \lambda}) = 0$ for all p dividing $\frac{N}{M}$, Lemma 3.3 implies that $\bar{\rho}_{f, \lambda}$ is ramified at all such p . Proposition 4.2 now applies to show that $\lambda \in \text{Cong}_N(f)$. Thus (9) holds, completing the proof. \square

Corollary 4.4. *If $\bar{\rho}_{f,\lambda}$ is absolutely irreducible, $\ell > 3$, and λ does not satisfy (1)–(9), then*

$$R_{\bar{\rho}_{f,\lambda}} \cong W(k_\lambda)[[T_1, T_2, T_3]].$$

Proof. This follows immediately from Theorem 4.3 and Corollary 2.4. \square \square

We also obtain the following partial converse to Theorem 4.3.

Theorem 4.5. *Assume that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible. Suppose that $\ell > 3$ and one of the following holds:*

- (1) $\ell \mid \varphi(N_S)$;
- (2) $a_p^2 \equiv (p+1)^2 p^{k-2} \omega(p) \pmod{\lambda}$ for some $p \mid \frac{N_S}{N}$, $p \neq \ell$;
- (3) λ is a congruence prime for f of level dividing $\frac{N}{p}$ for some $p \mid \frac{N}{M}$, $\ell \nmid p(p+1)$;
- (4) $k = 2$, $\ell \nmid N$, and $a_\ell^2 \equiv \omega(\ell) \pmod{\lambda}$.

Then $H^2(G_{\mathbf{Q}, S \cup \{\ell\}}, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) \neq 0$.

Proof. By Lemma 2.5 it suffices to show that these conditions guarantee that $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) \neq 0$ for some $p \in S$. If (1) holds, then $H^0(G_p, \bar{\varepsilon}_\ell) \neq 0$ for some $p \in S$, so that this is clear. If (2) holds, then by Lemma 3.1 we have $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}) \neq 0$. If (1) does not hold but (3) does hold, then Lemma 3.4 implies that $H^0(G_p, \bar{\varepsilon}_\ell \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}) \neq 0$. Finally, if (4) holds, then the proof of [21, Proposition 4.4] shows that $H^0(G_\ell, \bar{\varepsilon}_\ell \otimes \text{ad } \bar{\rho}_{f,\lambda}) \neq 0$. \square \square

5. EXAMPLES

In this section we use the data of [18] to bound the obstructed primes for the deformation problems associated to a few specific modular forms. Of course, the most interesting aspect of these computations are the determination of congruences between newforms. Using [18] we can check such congruences on the p^{th} Fourier coefficients for all $p < 1000$; by the results of [19] these checks are more than sufficient to prove that these congruences actually exist in our examples. We will not comment further on this issue.

For a modular form f , we let $\text{Red}(f)$ denote the set of primes λ of K such that $\bar{\rho}_{f,\lambda}$ is absolutely reducible. We recall the following well-known facts regarding $\text{Red}(f)$; see [5, Lemma 7.13] for example.

Lemma 5.1. *Let $f = \sum a_n q^n$ be a newform of weight k and level N with coefficient field K . Let λ be a prime of K dividing a rational prime ℓ . Suppose that $\lambda \in \text{Red}(f)$, so that*

$$\bar{\rho}_{f,\lambda} \otimes \bar{k}_\lambda \cong \chi_1 \oplus \chi_2$$

for characters $\chi_1, \chi_2 : G_{\mathbf{Q}} \rightarrow \bar{k}_\lambda^\times$. If ℓ does not divide N , then each χ_i has conductor dividing $N\ell$. If also $\ell > k$, then one of the χ_i has conductor dividing N , so that

$$a_p \equiv p^{k-1} + 1 \pmod{\lambda}$$

for all $p \equiv 1 \pmod{N}$.

In practice one uses the second condition to bound the set $\text{Red}(f)$ and the first condition to check each remaining λ not dividing N . For a prime λ dividing N , one can still check that $\bar{\rho}_{f,\lambda}$ is absolutely reducible, but it is much more difficult to show that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible; we will make no attempt to deal with this case below.

For a finite set of places S containing all places dividing $N\infty$, we let $\text{Obs}_S(f)$ denote the set of $\lambda \notin \text{Red}(f)$ such that

$$H^2(G_{\mathbf{Q}, S \cup \{\ell\}}, \varepsilon_\ell \otimes \text{ad } \bar{\rho}_{f, \lambda}) \neq 0,$$

or equivalently such that the deformation problem associated to

$$\bar{\rho}_{f, \lambda} : G_{\mathbf{Q}, S \cup \{\ell\}} \rightarrow \text{GL}_2 k_\lambda$$

is obstructed. We simply write $\text{Obs}(f)$ for $\text{Obs}_{\{p|N\infty\}}(f)$.

In the interests of space, we make the following notational conventions. Let K be a quadratic extension of \mathbf{Q} and fix a rational prime p . If p ramifies in K , then we simply write \mathfrak{p}_p for the prime of K above p . If p splits, then we will write \mathfrak{p}_p and $\bar{\mathfrak{p}}_p$ for the two primes of K above p , at least when it is not important to distinguish between them.

5.1. Weight 12, level 5, trivial character. There are three newforms of weight 12, level 5, and trivial character. The first has rational Fourier coefficients and q -expansion

$$f_1 = q + 34q^2 - 792q^3 - 892q^4 + 3125q^5 - 26928q^6 - 17556q^7 + \dots$$

while the other two,

$$\begin{aligned} f_2 = q &+ (-10 + 6\sqrt{151})q^2 + (-110 + 32\sqrt{151})q^3 + (3448 - 120\sqrt{151})q^4 \\ &- 3125q^5 + (30092 - 980\sqrt{151})q^6 + (28950 + 1056\sqrt{151})q^7 + \dots \end{aligned}$$

and its Galois conjugate, have field of Fourier coefficients $\mathbf{Q}(\sqrt{151})$. Note that $\text{Obs}(\bar{f}_2)$ is simply the set of conjugates of elements of $\text{Obs}(f_2)$, so that it suffices to study f_1 and f_2 .

Using Lemma 5.1, one computes that:

$$\text{Red}(f_1) = \{2, 5, 31\};$$

$$\text{Red}(f_2) = \{\mathfrak{p}_2, \mathfrak{p}_5, \bar{\mathfrak{p}}_5, (601, 358 + \sqrt{151})\}.$$

We now consider congruences. By comparing Fourier coefficients, one sees that f_1 and f_2 are congruent modulo primes above 2 and 5:

$$\text{Cong}_5(f_1) = \{2, 5\};$$

$$\text{Cong}_5(f_2) = \{\mathfrak{p}_2, (5, 1 + \sqrt{151})\}.$$

The only possible proper congruences is with the unique newform

$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \dots$$

of weight 12 and level 1; one computes that:

$$\text{Cong}_{<5}(f_1) = \{2, 29\};$$

$$\text{Cong}_{<5}(f_2) = \{\mathfrak{p}_2, (5, 4 + \sqrt{151}), (131, 46 + \sqrt{151})\}.$$

Both f_1 and f_2 are ordinary at 13, so that by Theorems 4.3 and 4.5 we conclude that:

$$\{29\} \subseteq \text{Obs}(f_1) \subseteq \{3, 7, 11, 13, 29\};$$

$$\{(131, 46 + \sqrt{151})\} \subseteq \text{Obs}(f_2) \subseteq \{\mathfrak{p}_3, \bar{\mathfrak{p}}_3, \mathfrak{p}_7, \bar{\mathfrak{p}}_7, (11), (13), (131, 46 + \sqrt{151})\}.$$

5.2. Weight 6, level 30, trivial character. There are two newforms of weight 6, level 30, and trivial character, both with rational Fourier coefficients:

$$f_1 = q + 4q^2 + 9q^3 - 16q^4 + 25q^5 + 36q^6 + 32q^7 - 192q^8 \\ - 162q^9 + 100q^{10} + 12q^{11} - 144q^{12} - 154q^{13} + \dots$$

$$f_2 = q - 4q^2 + 9q^3 - 16q^4 - 25q^5 - 36q^6 + 164q^7 + 192q^8 \\ - 162q^9 + 100q^{10} + 720q^{11} - 144q^{12} + 698q^{13} + \dots$$

Using Lemma 5.1, one computes that:

$$\text{Red}(f_1) = \{2, 3, 5\}; \\ \{2, 3\} \subseteq \text{Red}(f_2) \subseteq \{2, 3, 5\}.$$

The newforms f_1 and f_2 have a congruence modulo 12 (remember that one only checks the Fourier coefficients with exponent prime to 30), so that:

$$\text{Cong}_{30}(f_1) = \text{Cong}_{30}(f_2) = \{2, 3\}.$$

There are ten newforms of level dividing 30 and trivial character to consider for proper congruences. The most interesting occur for f_2 : it has a congruence modulo 19 with the newform

$$q + 7q^2 + 9q^3 + 17q^4 - 25q^5 + 63q^6 + 12q^7 - 105q^8 \\ - 162q^9 - 175q^{10} + 112q^{11} + 153q^{12} - 974q^{13} + \dots$$

of level 15, and modulo 31 with the newform

$$q - 4q^2 - 26q^3 - 16q^4 - 25q^5 + 104q^6 - 22q^7 + 192q^8 \\ + 433q^9 + 100q^{10} - 768q^{11} + 416q^{12} - 46q^{13} + \dots$$

of level 10. In any event, one computes:

$$\text{Cong}_{<30}(f_1) = \{2, 3, 5\}; \quad \text{Cong}_{<30}(f_2) = \{2, 3, 19, 31\}.$$

Both f_1 and f_2 are ordinary at 7, so that we conclude that:

$$\text{Obs}(f_1) \subseteq \{7\}; \\ \{19, 31\} \subseteq \text{Obs}(f_2) \subseteq \{5, 7, 19, 31\}.$$

To give an explicit example of an obstructed set, using (3) of Theorem 4.5 one finds that

$$\text{Obs}_{\{2,3,5,17,\infty\}}(f_1) = \{7\}.$$

5.3. Weight 3, level 35, character of conductor 7. There are four newforms of weight 3, level 35, and with quadratic character

$$\omega : (\mathbf{Z}/35\mathbf{Z})^\times \rightarrow \{\pm 1\}$$

the Legendre symbol $(\frac{\cdot}{7})$. All four are defined over $\mathbf{Q}(\sqrt{-5})$: two are

$$f_1 = q - q^2 - 2\sqrt{-5}q^3 - 3q^4 - \sqrt{-5}q^5 + 2\sqrt{-5}q^6 + 7q^7 + \dots$$

and its Galois conjugate while the other two are

$$f_2 = q + 2q^2 + \sqrt{-5}q^3 - \sqrt{-5}q^5 + 2\sqrt{-5}q^6 - (2 + 3\sqrt{-5})q^7 + \dots$$

and its Galois conjugate. As before, it suffices to study f_1 and f_2 . Using Lemma 5.1, one finds that:

$$\begin{aligned} \{\mathfrak{p}_2, \mathfrak{p}_3, \bar{\mathfrak{p}}_3, \mathfrak{p}_5\} &\subseteq \text{Red}(f_1) \subseteq \{\mathfrak{p}_2, \mathfrak{p}_3, \bar{\mathfrak{p}}_3, \mathfrak{p}_5, \mathfrak{p}_7, \bar{\mathfrak{p}}_7\} \\ \{\mathfrak{p}_3, \bar{\mathfrak{p}}_3\} &\subseteq \text{Red}(f_2) \subseteq \{\mathfrak{p}_3, \bar{\mathfrak{p}}_3, \mathfrak{p}_5, \mathfrak{p}_7, \bar{\mathfrak{p}}_7\}. \end{aligned}$$

The newforms f_1 and f_2 have a congruence modulo 3, while f_1 and \bar{f}_2 have no congruences; thus:

$$\text{Cong}_{35}(f_1) = \text{Cong}_{35}(f_2) = \{\mathfrak{p}_3, \bar{\mathfrak{p}}_3\}.$$

Since ω has conductor 7, the only proper congruences we need to check are with the unique newform

$$q - 3q^2 + 5q^4 - 7q^7 - 3q^8 - 9q^9 - 6q^{11} + \dots$$

of weight 3, level 7, and character $(\frac{\cdot}{7})$. One finds that:

$$\text{Cong}_{<35}(f_1) = \{\mathfrak{p}_2\};$$

$$\text{Cong}_{<35}(f_2) = \{\mathfrak{p}_5\}.$$

Theorem 4.3 allows us to conclude that:

$$\text{Obs}(f_1) \subseteq \{\mathfrak{p}_7, \bar{\mathfrak{p}}_7\}$$

$$\text{Obs}(f_2) \subseteq \{\mathfrak{p}_2, \mathfrak{p}_5, \mathfrak{p}_7, \bar{\mathfrak{p}}_7\}.$$

REFERENCES

- [1] Nigel Boston and Stephon Ullom, *Representations related to CM elliptic curves*, Math. Proc. Cambridge Philos. Soc. 113 (1993) 71–85.
- [2] Henri Carayol, *Sur les représentations l -adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986) 409–468.
- [3] Bart de Smit and Hendrik Lenstra, Explicit construction of universal deformation rings, in: G. Cornell, J. Silverman and G. Stevens, eds., *Modular forms and Fermat's last theorem* (Springer, New York, 1997) 313–326.
- [4] Fred Diamond, An extension of Wiles' results, in: G. Cornell, J. Silverman and G. Stevens, eds., *Modular forms and Fermat's last theorem* (Springer, New York, 1997) 475–489.
- [5] Fred Diamond, Matthias Flach, and Li Guo, *Adjoint motives of modular forms and the Tamagawa number conjecture*, preprint.
- [6] Fred Diamond and Richard Taylor, *Lifting modular mod ℓ representations*, Duke Math. J. **74** (1994) 253–269.
- [7] Mark Dickinson, A criterion for existence of a universal deformation ring, in: *Arithmetic algebraic geometry (Park City, UT, 1999)*, 233–406, Amer. Math. Soc., Providence, RI, 2001.
- [8] Bas Edixhoven, Serre's conjecture, G. Cornell, J. Silverman and G. Stevens, eds., *Modular forms and Fermat's last theorem* (Springer, New York, 1997) 475–489.
- [9] Matthias Flach, *A generalisation of the Cassels-Tate pairing*, J. Reine Angew. Math. **412** (1990) 113–127.
- [10] Matthias Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. **109** (1992) 307–327.
- [11] Eknath Ghate, *Adjoint L -values and primes of congruence for Hilbert modular forms*, Compositio Math. **132** (2002) 243–281.
- [12] Haruzo Hida, *Congruence of cusp forms and special values of their zeta functions*, Invent. Math. **63** (1981) 225–261.
- [13] ———, *On congruence divisors of cusp forms as factors of the special values of their zeta functions*, Invent. Math. **64** (1981) 221–262.
- [14] Barry Mazur, Deforming Galois representations, in: Y. Ihara, K. Ribet and J.-P. Serre, eds., *Galois groups over \mathbf{Q}* (Springer, New York, 1989) 385–437.
- [15] James Milne, *Arithmetic duality theorems* (Academic Press Inc., Boston, MA, 1986).

- [16] Kenneth Ribet, Galois representations attached to eigenforms with nebentypus, in: *Modular functions of one variable, V (Bonn, 1976)*, Lecture Notes in Math. **601** (Springer, Berlin, 1977) 17–51.
- [17] Kenneth Ribet, *Mod p Hecke operators and congruences between modular forms*, Invent. Math. **71** (1983) 193–205.
- [18] William Stein, *The modular forms explorer*, available at <http://modular.fas.harvard.edu/mfd/mfe>.
- [19] Jacob Sturm, On the congruence of modular forms, in: *Number theory (New York, 1984–1985)*, Lecture Notes in Math. **1240** (Springer, Berlin, 1987) 275–280.
- [20] Tom Weston, An overview of a theorem of Flach, in: B. Conrad and K. Rubin, eds., *Arithmetic algebraic geometry* (Amer. Math. Soc., Providence, RI, 2001) 345–375.
- [21] ———, *Unobstructed modular deformation problems*, to appear in American J. of Math.
- [22] Atsushi Yamagami, *On the unobstructedness of the deformation problems of residual modular representations*, preprint.