LECTURE 4: SYMPLECTIC GROUP ACTIONS

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1. Symplectic circle actions

We set $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ throughout.

Let $(M, \omega)$ be a symplectic manifold. A symplectic $S^1$-action on $(M, \omega)$ is a smooth family $\psi_t \in \text{Symp}(M, \omega)$, $t \in S^1$, such that $\psi_{t+s} = \psi_t \circ \psi_s$ for any $t, s \in S^1$. One can easily check that the corresponding vector fields $X_t \equiv \frac{d}{dt} \psi_t \circ \psi_t^{-1}$ is time-independent, i.e., $X_t = X$ is constant in $t$. We call $X$ the associated vector field of the given symplectic $S^1$-action.

Note that $X$ is a symplectic vector field, i.e., $X(\omega) = \text{closed}$. When $X(\omega) = dH$ is an exact 1-form, the corresponding symplectic $S^1$-action is called a Hamiltonian $S^1$-action, and the function $H : M \to \mathbb{R}$ is called a moment map. Note that $H$ is uniquely determined up to a constant. We point out that a symplectic $S^1$-action on $(M, \omega)$ is automatically Hamiltonian if $H^1(M; \mathbb{R}) = 0$.

In general, $X(\omega)$ is only a closed form. In this case, one can perturb $\omega$ by adding a sufficiently small $S^1$-invariant harmonic 2-form $\beta$ so that $\omega + \beta$ is still symplectic and $S^1$-invariant, and furthermore, the deRham cohomology class $[\omega + \beta]$ lies in $H^2(M; \mathbb{Q})$. A positive integral multiple of $\omega + \beta$, $\omega' = N(\omega + \beta)$, is a $S^1$-invariant symplectic form on $M$ such that the deRham cohomology class of $\iota(X)\omega'$ lies in $H^1(M; \mathbb{Z})$. Consequently, there exists a smooth function $H : M \to \mathbb{R}/\mathbb{Z}$ such that $\iota(X)\omega' = dH$. Such a circle-valued smooth function $H : M \to \mathbb{R}/\mathbb{Z}$ is called a generalized moment map. As far as the topology of the $S^1$-action is concerned, one may always assume that there is a generalized moment map.

Let $H$ be a moment map or generalized moment map of a given symplectic $S^1$-action. We shall make the following two observations.

1. Each level surface $H^{-1}(\lambda)$ is invariant under the $S^1$-action, because $dH(X) = \omega(X, X) = 0$ (where $X$ is the associated vector field of the given symplectic $S^1$-action).

2. A point $p \in M$ is a fixed point of the $S^1$-action if and only if $p$ is a critical point of $H$, i.e., $dH = 0$ at $p$. This is because $p \in M$ is a fixed point of the $S^1$-action iff $X = 0$ at $p$ iff $dH = \iota(X)\omega = 0$ at $p$.

Now consider a level surface $H^{-1}(\lambda)$ where $\lambda$ is a regular value of $H$. Then $H^{-1}(\lambda)$ is a hypersurface in $M$ which does not contain any fixed points of the $S^1$-action. When the $S^1$-action is free on $H^{-1}(\lambda)$, the quotient $B_\lambda \equiv H^{-1}(\lambda)/S^1$ is a smooth manifold. In general, the $S^1$-action may have finite isotropy on $H^{-1}(\lambda)$, and the quotient $B_\lambda \equiv H^{-1}(\lambda)/S^1$ is a smooth orbifold. In any case, one observes that $\dim B_\lambda = \dim M - 2$. 
The next proposition shows that there exists a natural symplectic structure $\omega_\lambda$ on $B_\lambda$. The symplectic manifold (or orbifold) $(B_\lambda, \omega_\lambda)$ is called the symplectic quotient or the reduced space at $\lambda$. The process of going from $(M, \omega)$ to $(B_\lambda, \omega_\lambda)$ is called symplectic reduction.

**Proposition 1.1.** There exists a canonically defined symplectic structure $\omega_\lambda$ on $B_\lambda$ such that $\pi^*\omega_\lambda = \omega|_{H^{-1}(\lambda)}$, where $\pi : H^{-1}(\lambda) \to H^{-1}(\lambda)/S^1 \equiv B_\lambda$ is the projection.

**Proof.** For simplicity, we assume the $S^1$-action on $H^{-1}(\lambda)$ is free, and consequently $B_\lambda$ is a smooth manifold. To simplify the notation, we set $Q = H^{-1}(\lambda)$.

Let’s recall a basic result about differentiable Lie group actions on manifolds — the existence of local slice. In the present situation, the result amounts to say that for any point $q \in Q$, there exists a submanifold $O_q$ of codimension 1 containing $q$, such that $S^1 \times O_q$ embeds into $Q$ $S^1$-equivariantly. $O_q$ is called a local slice at $q$, and the set $\{O_q|q \in Q\}$ forms an atlas of charts for the differentiable structure on the quotient $Q/S^1$. If $q' \in Q$ lies in $S^1 \times O_q$, then there is a local diffeomorphism $\phi_{qq'}$ from a neighborhood $U$ of $q'$ in the slice $O_{q'}$ into $O_q$ and a function on $U$ into $S^1$, $f_{qq'}$, such that $U \subset O_{q'}$ may be identified with the graph of $f_{qq'}$ over the image of $\phi_{qq'}$ in $S^1 \times O_q$. Note that with the differentiable structure on the quotient $Q/S^1 = B_\lambda$ described above, the projection $\pi : Q \to B_\lambda$ becomes a principal $S^1$-bundle over $B_\lambda$, with local trivializations of the bundle given by projections $S^1 \times O_q \to O_q$, $q \in Q$.

The symplectic structure $\omega_\lambda$ is defined by pulling-back $\omega$ to each local slice $O_q$. This definition immediately gives the closedness of $\omega_\lambda$ as well as the equation $\pi^*\omega_\lambda = \omega|_Q$. To see that $\omega_\lambda$ is well-defined, i.e., the pull-back of $\omega$ to each local slice can be patched up, we note that the local slices are graphs over each other locally, and that the tangent direction of $S^1$ in $S^1 \times O_q$ lies in $TQ\omega$ at each point. The nondegeneracy of $\omega_\lambda$ follows from the fact that $\dim T_qQ^\omega = \dim T_qM - \dim T_qQ = 1$, so that $T_qQ^\omega$ is actually generated by the tangent direction of $S^1$.

**Example 1.2.** (Product of $S^1$-actions). For $j = 1, 2$, let $(M_j, \omega_j)$ be a symplectic manifold with a symplectic $S^1$-action $t \mapsto \psi_t^j$, $t \in S^1$. Then for any $m_1, m_2 \in Z$ such that $\gcd(m_1, m_2) = 1$, there is a canonical $S^1$-action on the product $(M_1 \times M_2, \omega_1 \times \omega_2)$, $t \mapsto \psi_t$, $t \in S^1$, where $\psi_t = \psi_{m_1 t}^1 \times \psi_{m_2 t}^2$. Moreover, if $H_1$, $H_2$ are moment maps of the $S^1$-actions $\psi_t^1$, $\psi_t^2$ respectively, then $H = m_1 H_1 + m_2 H_2$ is a moment map of the product $\psi_t$. To see this, note that if $X_j$, $j = 1, 2$, is the vector field on $M_j$ which generates the $S^1$-action $\psi_t^j$, then $X = \langle m_1 X_1, m_2 X_2 \rangle$ is the vector field on $M_1 \times M_2$ which generates the $S^1$-action $\psi_t$. The claim about the moment maps follows immediately from $\iota(X)(\omega_1 \times \omega_2) = m_1 \iota(X_1)\omega_1 + m_2 \iota(X_2)\omega_2$.

**Example 1.3.** (Holomorphic $S^1$-actions on Kähler manifolds). For any holomorphic $S^1$-action on a Kähler manifold, one can choose an invariant Kähler metric, so that the $S^1$-action becomes a symplectic $S^1$-action with respect to the invariant Kähler form.

**Example 1.4.** Consider $(\mathbb{R}^2, \omega_0)$ with symplectic $S^1$-action given by the complex multiplication $z \mapsto e^{it}z$, $t \in S^1$. Here we identify $\mathbb{R}^2 = \mathbb{C}$. To determine the moment map, we note that the $S^1$-action is generated by the vector field $X = -y\partial x + x\partial y$. With this we see the moment map is given by $H(z) = -\frac{1}{2}|z|^2$, because $\iota(X)\omega_0 = -ydy - xdx$. 

Now for any $\mathbf{m} = (m_0, m_1, \ldots, m_n)$ where each $m_i \in \mathbb{Z}$ and $gcd(m_0, m_1, \ldots, m_n) = 1$, consider more generally the symplectic $S^1$-action on $(\mathbb{R}^{2n+2}, \omega_0)$, which is defined by

$$(z_0, z_1, \ldots, z_n) \mapsto (e^{im_0 t} z_0, e^{im_1 t} z_1, \ldots, e^{im_n t} z_n), \ t \in S^1.$$ 

By Example 1.2, the moment map of the $S^1$-action is

$$H(z_0, z_1, \ldots, z_n) = -\frac{1}{2} (m_0 |z_0|^2 + m_1 |z_1|^2 + \cdots + m_n |z_n|^2).$$

For the special case where $\mathbf{m} = (1, 1, \ldots, 1)$, the level surface $H^{-1}(\lambda)$, $\lambda < 0$, is the $(2n + 1)$-dimensional sphere of radius $-2\lambda$, and the $S^1$-action on $H^{-1}(\lambda)$ is given by the Hopf fibration. The reduced space $(B_\lambda, \omega_\lambda)$ at $\lambda = -\frac{1}{2}$ is $\mathbb{CP}^n$ with $\omega_\lambda$ being $\pi$ times the Fubini-Study form $\omega_0$ on $\mathbb{CP}^n$, see Example 1.9 in Lecture 1.

**Example 1.5.** Note that for any $\mathbf{m} = (m_0, m_1, \ldots, m_n)$, the corresponding symplectic $S^1$-action on $(\mathbb{R}^{2n+2}, \omega_0)$ with weights $\mathbf{m}$ preserves the unit sphere $S^{2n+1}$ and commutes with the Hopf fibration. Hence there is an induced $S^1$-action on $\mathbb{CP}^n$, which must be symplectic with respect to the Fubini-Study form and has the moment map

$$H(z_0, z_1, \ldots, z_n) = -\frac{1}{2} (m_0 |z_0|^2 + m_1 |z_1|^2 + \cdots + m_n |z_n|^2)$$

where $|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1$, due to the relation $\pi^* \omega_\lambda = \omega|_{H^{-1}(\lambda)}$ between the symplectic forms in the symplectic reduction given in Proposition 1.1, and the fact that the vector field on $\mathbb{CP}^n$ which generates the induced $S^1$-action is the push-down of the corresponding vector field on $S^{2n+1}$ under the Hopf fibration $\pi : S^{2n+1} \rightarrow \mathbb{CP}^n$.

In terms of the homogeneous coordinates $z_0, z_1, \ldots, z_n$ on $\mathbb{CP}^n$, the $S^1$-action is given by

$$[z_0, z_1, \ldots, z_n] \mapsto [e^{im_0 t} z_0, e^{im_1 t} z_1, \ldots, e^{im_n t} z_n], \ t \in S^1.$$ 

The corresponding moment map is given by

$$H([z_0, z_1, \ldots, z_n]) = -\frac{1}{2} \sum_{j=0}^n \frac{1}{|z_j|^2} (m_0 |z_0|^2 + m_1 |z_1|^2 + \cdots + m_n |z_n|^2).$$

Note that in order for the $S^1$-action on $\mathbb{CP}^n$ to be effective, one needs to impose additional conditions

$$gcd(m_0 - m_j, \ldots, m_n - m_j) = 1, \ \forall j = 0, \ldots, n.$$

On the other hand, for any $m \in \mathbb{Z}$, the weights $\mathbf{m} - m \equiv (m_0 - m, m_1 - m, \ldots, m_n - m)$ defines the same $S^1$-action on $\mathbb{CP}^n$ as the weights $\mathbf{m} = (m_0, m_1, \ldots, m_n)$. Note that the corresponding moment maps change by a constant $\frac{m}{2}$.

Consider the case where $n = 1$. The above construction gives rise to symplectic $S^1$-actions on $\mathbb{CP}^1 = S^2$, where different choices of weights $\mathbf{m}$ yield the same $S^1$-action. If we let $\mathbf{m} = (0, -1)$ and identify $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, the $S^1$-action is simply given by the complex multiplication on $\mathbb{C}$, which has moment map

$$H(z) = \frac{1}{2} \cdot \frac{1}{1 + |z|^2}.$$ 

The fixed points are $\{0, \infty\}$, and the corresponding critical values of the moment map are $H(0) = \frac{1}{2}$, $H(\infty) = 0$. The only difference between this example of a Hamiltonian
Proposition 1.7. A moment map (or generalized moment map) of a symplectic $\mathbb{S}^1$-action is Morse-Bott with the following additional properties: (1) the critical submanifolds are symplectic submanifolds, (2) the index at each critical point is always an even number.

Proof. The proposition follows from an equivariant version of the Darboux theorem, which also gives a model for the moment map near a critical point. Since the problem is local, there is no difference for the case of generalized moment map.

Let $(M, \omega)$ be a symplectic manifold with a symplectic $\mathbb{S}^1$-action $\psi_t$, $t \in \mathbb{S}^1$. The equivariant version of Darboux theorem states as follows: for any fixed point $p \in M$, there is a chart $\phi : \mathbb{R}^n \rightarrow M$ centered at $p$ such that

$$\phi^* \omega = \omega_0,$$

where $\phi^* \omega$ is the standard symplectic form on $\mathbb{R}^n$, and the pull-back $\mathbb{S}^1$-action $\phi^* \psi_t \equiv \phi^{-1} \circ \psi_t \circ \phi$ on $\mathbb{R}^n$ is linear and is given under the standard identification $\mathbb{R}^n = \mathbb{C}^n$ by

$$(z_1, z_2, \ldots, z_n) \mapsto (e^{im_1 t} z_1, e^{im_2 t} z_2, \ldots, e^{im_n t} z_n), \quad t \in \mathbb{S}^1,$$

for some $m_1, m_2, \ldots, m_n \in \mathbb{Z}$ with $\gcd(m_1, m_2, \ldots, m_n) = 1$.

Let $H$ be a moment map of the symplectic $\mathbb{S}^1$-action on $M$. Then the equivariant Darboux theorem together with Example 1.4 implies that there exists a chart $\phi : \mathbb{R}^{2n} \rightarrow M$ centered at $p$ such that

$$\phi^* H(x_1, \ldots, x_n, y_1, \ldots, y_n) = -\frac{1}{2} [m_1 (x_1^2 + y_1^2) + \cdots + m_n (x_n^2 + y_n^2)] + H(p)$$

for some $m_1, m_2, \ldots, m_n \in \mathbb{Z}$ with $\gcd(m_1, m_2, \ldots, m_n) = 1$. With a further change of coordinates it follows immediately that $H$ is a Morse-Bott function. The index at $p$ is twice of the number of positive $m_j$’s, hence is always an even number. The critical submanifold is locally defined near $p$ by the equations $x_j = y_j = 0$ for all $j$ with $m_j \neq 0$, hence is symplectic.

Next we sketch a proof of the equivariant Darboux theorem. We pick an $\mathbb{S}^1$-invariant metric on $M$, and by Theorem 1.15 of Lecture 2 (parametric version) it gives rise to an $\mathbb{S}^1$-invariant $J \in \mathcal{J}(M, \omega)$. Note that the Hermitian metric on $(M, J)$, $g_J(\cdot, \cdot) \equiv \omega(\cdot, J \cdot)$, is also $\mathbb{S}^1$-invariant.
Let $p \in M$ be a fixed point. Then the induced $\mathbb{S}^1$-action on $T_p M$, which is naturally a Hermitian vector space with $J$ and $g_J$, has the form
\[
\sum_{j=1}^{n} z_j u_j \mapsto \sum_{j=1}^{n} e^{im_j t} z_j u_j, \quad t \in \mathbb{S}^1,
\]
with respect to a unitary basis $u_1, u_2, \cdots, u_n$, where $m_j \in \mathbb{Z}$, $j = 1, 2, \cdots, n$, obeys $gcd(m_1, m_2, \cdots, m_n) = 1$.

We define a chart $\psi : \mathbb{R}^{2n} \to M$ centered at $p$ by
\[
\psi : (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto \exp_p \left( \sum_{j=1}^{n} (x_j + iy_j) u_j \right),
\]
where the exponential map $\exp_p$ is with respect to the metric $g_J$. Note that with respect to the linear $\mathbb{S}^1$-action on $\mathbb{R}^{2n}$ (identified with $T_p M$ via the unitary basis $u_1, u_2, \cdots, u_n$), the local diffeomorphism $\psi$ is $\mathbb{S}^1$-equivariant because the metric $g_J$ is.

Now consider the pull-back symplectic form on $\mathbb{R}^{2n}$, $\psi^* \omega$. Both $\psi^* \omega$ and $\omega_0$ are $\mathbb{S}^1$-invariant with respect to the linear $\mathbb{S}^1$-action on $\mathbb{R}^{2n}$, and $\psi^* \omega = \omega_0$ at the origin $0 \in \mathbb{R}^{2n}$. The proof of Lemma 1.1 in Lecture 3 works perfectly in an equivariant context, from which the equivariant Darboux theorem follows.

Note that with respect to the metric $g_J$ (for any $J \in \mathfrak{J}(M, \omega)$) the gradient vector field for a moment map $H$ is given by $\text{grad} H = JX$, where $X$ is the vector field which generates the $\mathbb{S}^1$-action.

The fact that a moment map of a symplectic $\mathbb{S}^1$-action is Morse-Bott with even index at each critical point has the following corollary by a standard argument in Morse theory.

**Corollary 1.8.** Let $H : M \to \mathbb{R}$ be a moment map of a Hamiltonian $\mathbb{S}^1$-action on a compact, connected manifold. Then each level surface $H^{-1}(\lambda)$ is connected. In particular, the critical submanifolds at the maximal and minimal values of $H$ are connected.

The standard model for a moment map $H$ near a critical point as we obtained in the proof of Proposition 1.7 allows one to explicitly analyzing the change of the topology of the reduced spaces passing a critical value of the moment map in terms of the weights $m_j$ at each critical point in the pre-image of that critical value. On the other hand, for any interval $I$ of regular values, Morse theory allows one to identify $H^{-1}(I)$ diffeomorphically with the product $H^{-1}(\lambda_0) \times I$ for any $\lambda_0 \in I$ (e.g. using the gradient flow of $H$), which can be made $\mathbb{S}^1$-equivariantly. The next proposition describes the relation between the symplectic form $\omega$ on $H^{-1}(I)$ and the reduced spaces $(B_\lambda, \omega_\lambda)$, $\lambda \in I$, and the first Chern class of the $\mathbb{S}^1$-principal bundle $\pi : H^{-1}(\lambda_0) \to B_{\lambda_0}$. For simplicity we assume that the $\mathbb{S}^1$-action on $H^{-1}(I)$ is free, so that each reduced space $B_\lambda$, $\lambda \in I$, is a smooth manifold. Note that all $H^{-1}(\lambda)$, $B_\lambda$, $\lambda \in I$, are diffeomorphic; we denote the underlying manifolds by $P$, $B$ respectively. (We warn that the $\mathbb{S}^1$-action on each $H^{-1}(\lambda)$, $\lambda \in I$, is assumed to be on the left, so when $H^{-1}(\lambda)$ is regarded as a $\mathbb{S}^1$-principal bundle, where the action is always assumed to be on the right, we mean
the conjugate \( S^1 \)-action. For example, in this way the first Chern class of the Hopf fibration \( S^3 \to \mathbb{C}P^1 \) evaluates positively on the fundamental class of \( \mathbb{C}P^1 \).

**Proposition 1.9.** (1) Let \( c \in H^2(B; \mathbb{Z}) \) be the first Chern class of the \( S^1 \)-principal bundle \( \pi : P \to B \) and let \( \{ \omega_\lambda | \lambda \in I \} \) be a smooth family of symplectic forms on \( B \) such that their deRham cohomology classes satisfy

\[
[\omega_\lambda] = [\omega_{\mu}] - 2\pi(\lambda - \mu) \cdot c.
\]

There is an \( S^1 \)-invariant symplectic form \( \omega \) on \( P \times I \) with a moment map \( H \) equal to the projection \( P \times I \to I \) and with reduced spaces \( (B, \omega_\lambda) \), \( \lambda \in I \).

(2) Conversely, every \( S^1 \)-invariant symplectic form \( \omega \) arises in the above way. Moreover, up to \( S^1 \)-equivariant symplectomorphisms such a \( S^1 \)-invariant symplectic form on \( P \times I \) is uniquely determined by the family of symplectic forms \( \{ \omega_\lambda | \lambda \in I \} \) on \( B \).

**Proof.** (1) Since the deRham cohomology class of \( \frac{d}{d\lambda} \omega_\lambda \) represents \( -2\pi c \), there exists a smooth family of imaginary valued 1-forms \( A_\lambda \) on \( P \), i.e., the connection 1-forms, such that \( \frac{1}{2\pi} dA_\lambda = -\frac{1}{\pi} \pi^* \frac{d}{d\lambda} \omega_\lambda \). Let \( X \) be the vector field which generates the \( S^1 \)-action. Then \( A_\lambda(X) = -i \) because as we remarked before \( P \) is regarded as an \( S^1 \)-principal bundle on \( B \) with the conjugate action. Set \( \alpha_\lambda = iA_\lambda \). Then \( \alpha_\lambda(X) = 1 \), and \( \pi^* \frac{d}{d\lambda} \omega_\lambda + d\alpha_\lambda = 0 \). With these understood,

\[
\omega = \pi^* \omega_\lambda + \alpha_\lambda \wedge d\lambda
\]

is an \( S^1 \)-invariant symplectic form on \( P \times I \) with a moment map \( H \) equal to the projection \( P \times I \to I \) and with reduced spaces \( (B, \omega_\lambda) \), \( \lambda \in I \).

(2) Note that as a 2-form on \( P \times I \), \( \omega \) may be written as

\[
\omega = \beta_\lambda + \alpha_\lambda \wedge d\lambda
\]

for some \( \alpha_\lambda \in \Omega^1(P) \) and \( \beta_\lambda \in \Omega^2(P) \). Let \( X \) be the vector field which generates the \( S^1 \)-action. Since the moment map of the \( S^1 \)-action is the projection \( P \times I \to I \), we see that \( \iota(X)\beta_\lambda = 0 \) and \( \alpha_\lambda(X) = 1 \). The former implies that \( \beta_\lambda \) descents to a smooth family of 2-forms \( \omega_\lambda \) on \( B \), so that \( \pi^* \omega_\lambda = \beta_\lambda \). The nondegeneracy of \( \omega \) implies that each \( \omega_\lambda \) is nondegenerate, and the closedness of \( \omega \) implies

\[
d\omega_\lambda = 0, \quad \frac{d}{d\lambda} \beta_\lambda + d\alpha_\lambda = 0.
\]

Note that \( -i\alpha_\lambda \) are connection 1-forms on \( P \), so that the first Chern class \( c \) is represented by \( \frac{i}{2\pi} d(-i\alpha_\lambda) = \frac{1}{2\pi} d\alpha_\lambda \). This gives the relation

\[
[\omega_\lambda] = [\omega_{\mu}] - 2\pi(\lambda - \mu) \cdot c.
\]

For the uniqueness, consider two different such symplectic forms

\[
\omega = \pi^* \omega_\lambda + \alpha_\lambda \wedge d\lambda, \quad \omega' = \pi^* \omega_\lambda + \alpha'_\lambda \wedge d\lambda.
\]

One can form a smooth family of such symplectic forms

\[
\omega_t = \pi^* \omega_\lambda + ((1-t)\alpha_\lambda + t\alpha'_\lambda) \wedge d\lambda, \quad t \in [0, 1].
\]

The uniqueness follows from an equivariant version of Moser’s stability theorem applied to \( \omega_t \). We leave it as an exercise. 

\[ \square \]
Example 1.10. (1) Consider the $S^1$-action on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ at the end of Example 1.5, which is given by the complex multiplication of $e^{it}$ and has moment map $H(z) = 1/2(1 + |z|^2)$. Since in this case the reduced spaces are a single point, $\omega_{\lambda} = 0$, and therefore the symplectic form

$$\omega = \alpha_\lambda \wedge d\lambda.$$ 

In the polar coordinates $(r, \theta)$ on $\mathbb{C}$, $\alpha_\lambda = d\theta$, so that

$$\omega = d\theta \wedge d\left(\frac{1}{2(1 + r^2)}\right) = \frac{rdr \wedge d\theta}{(1 + r^2)^2} = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}.$$ 

Direct calculation shows

$$\int_{\mathbb{C}P^1} \omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} = \pi$$

as we claimed.

(2) Consider the $S^1$-action on $(\mathbb{R}^{2n+2}, \omega_0)$ in Example 1.4 with weights $m = (1, 1, \cdots, 1)$. The moment map is

$$H(z_0, z_1, \cdots, z_n) = -\frac{1}{2}(|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2).$$

For any $\lambda < 0$ the level surface $H^{-1}(\lambda)$ is the sphere of radius $-2\lambda$, and the $S^1$-action on $H^{-1}(\lambda)$ is given by the Hopf fibration, with the quotient being $\mathbb{C}P^n$. We have claimed in Example 1.4 that the symplectic form $\omega_{\lambda}$ at $\lambda = -\frac{1}{2}$ is $\pi$ times the Fubini-Study form on $\mathbb{C}P^n$, which is normalized so that the integral of its $n$-th power over $\mathbb{C}P^n$ equals 1. Note that this implies that $\int_{\mathbb{C}P^n} \omega_{-1/2}^n = \pi^n$. We shall next give an independent verification of this fact using Proposition 1.9.

First note that as $\lambda \to 0$ the form $\omega_{\lambda}$ converges to 0. This gives, by Proposition 1.9,

$$[\omega_{-\frac{1}{2}}] = 0 - 2\pi(-\frac{1}{2} - 0) \cdot c = \pi \cdot c,$$

where $c$ is the first Chern class of the Hopf fibration. It is known that $c \in H^2(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}$ is the positive generator, so that $e^\pi |\mathbb{C}P^n| = 1$. This implies that

$$\int_{\mathbb{C}P^n} \omega_{-\frac{1}{2}}^n = \pi^n$$

as we claimed.

(3) Consider the $S^1$-action on $\mathbb{C}P^2$ in Example 1.5 with weights $m = (0, -1, -2)$. There are three fixed points $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$, where the moment map has values 0, $\frac{1}{2}$ and 1 respectively. Using the standard model for the moment map near a critical point as in the proof of Proposition 1.7, it is easy to check that for any regular value $\lambda$, the reduced space $B_\lambda$ is the weighted projective space $\mathbb{C}P^1(1, 2)$, which is the quotient space $(\mathbb{C}^2 \setminus \{(0, 0)\})/\sim$, where $(z_1, z_2) \sim (lz_1, z_2 z_l)$. (Note that $\mathbb{C}P^1(1, 2)$ is a 2-dimensional orbifold, with one singular point of order 2.) However, for $\lambda \in (0, \frac{1}{2})$, the first Chern class of the (orbifold) $S^1$-principal bundle $H^{-1}(\lambda) \to B_\lambda$ equals $-\frac{1}{2} \in H^2(\mathbb{C}P^1(1, 2); \mathbb{Q})$ and for $\lambda \in (\frac{1}{2}, 1)$, it equals $\frac{1}{2} \in H^2(\mathbb{C}P^1(1, 2); \mathbb{Q})$. Note that the first Chern class changes by 1 when passing the critical value $\lambda = \frac{1}{2}$. 


For a Hamiltonian $S^1$-action with at most isolated fixed points, the moment map is a Morse function. Propositions 1.7 and 1.9 show that the weights of the induced action on the tangent space of each fixed point contain vital information about the equivariant symplectic geometry of the manifold. There are certain constraints amongst the weights of the fixed points, as shown in the following beautiful theorem of Duistermaat and Heckman.

**Theorem 1.11.** (Duistermaat-Heckman). Assume a Hamiltonian $S^1$-action on a compact $2n$-dimensional symplectic manifold $(M, \omega)$ has only isolated fixed points. Let $H : M \to \mathbb{R}$ be a moment map, and let $e(p)$ denote the product of the weights at a fixed point $p$. Then

$$\int_M e^{-2\pi hH} \omega^n = \sum_p \frac{e^{-2\pi hH(p)}}{h^n e(p)}$$

for every $h \in \mathbb{C}$, where the sum on the right-hand side runs over all fixed points of the $S^1$-action.

**Example 1.12.** If one expands both sides of the Duistermaat-Heckman formula as power series in $h$ and then compares the coefficients, the following set of constraints are obtained:

$$\sum_p \frac{H(p)^k}{e(p)} = 0, \text{ for } k = 0, 1, \ldots, n-1,$$

and

$$\int_M \omega^n = (-2\pi)^n \sum_p \frac{H(p)^n}{e(p)}.$$

We shall check this out on an example of $S^1$-action on $\mathbb{C}P^2$ as discussed in Example 1.5, with weights $m = (0, -2, -5)$. It is easy to check that there are three isolated fixed points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, $p_3 = [0, 0, 1]$, which have weights $(-2, -5)$, $(2, -3)$, and $(5, 3)$ respectively. Let $H$ be the standard moment map as given in Example 1.5. Then $H(p_1) = 0$, $H(p_2) = 1$, and $H(p_3) = \frac{5}{2}$. We also know (see Example 1.10 (2)) that $\int_{\mathbb{C}P^2} \omega^2 = \pi^2$. With the preceding understood, the set of constraints obtained from the Duistermaat-Heckman formula are the following for this example:

$$\frac{1}{(-2) \cdot (-5)} + \frac{1}{2 \cdot (-3)} + \frac{1}{5 \cdot 3} = 0, \quad \frac{0}{(-2) \cdot (-5)} + \frac{1}{2 \cdot (-3)} + \frac{5}{2 \cdot 3} = 0$$

and

$$(-2\pi)^2 \left(\frac{0^2}{(-2) \cdot (-5)} + \frac{1^2}{2 \cdot (-3)} + \frac{(\frac{5}{2})^2}{5 \cdot 3}\right) = \pi^2.$$

We end with a discussion on the question as to which symplectic $S^1$-actions on a compact closed manifold are Hamiltonian. Note that a necessary condition is that the $S^1$-action must have a fixed point, because the fixed points correspond to the critical points of a moment map, and a smooth ($\mathbb{R}$-valued) function on a compact manifold must have a critical point.

We will give a sufficient condition utilizing the following lemma.
**Lemma 1.13.** Let $(M, \omega)$ be a compact closed symplectic manifold equipped with a symplectic $S^1$-action. We assume without loss of generality that $\int_M \omega^n = 1$. Denote by $\alpha$ the homology class of an orbit of the $S^1$-action in $H_1(M; \mathbb{R})$. Then $\alpha$ is Poincaré dual to $2\pi \cdot [\iota(X) \omega^n] \in H^{2n-1}(M; \mathbb{R})$, where $X$ is the vector field generating the $S^1$-action.

**Proof.** Let $\psi_t, t \in S^1$, be the $S^1$-action and let $\gamma(t) = \psi_t(q), q \in M$, be a non-constant orbit in $M$. We choose a volume form $\sigma \in \Omega^{2n}(M)$ which is supported in a small neighborhood of $\gamma$ and satisfies $\int_M \sigma = 1$. By averaging over $S^1$ we may assume that $\psi_t^* \sigma = \sigma$. Since by assumption $\int_M \omega^n = 1$, we see that $\sigma$ and $\omega^n$ are cohomologous, and $\sigma - \omega^n = d\beta$ for some $\beta \in \Omega^{2n-1}(M)$, which, by averaging over $S^1$, may be assumed to satisfy $\psi_t^* \beta = \beta$ also. Now we have

$$\iota(X) \sigma - \iota(X) \omega^n = \iota(X) d\beta = \mathcal{L}_X \beta - d(\iota(X) \beta) = d(-\iota(X) \beta),$$

so that $\iota(X) \sigma$ and $\iota(X) \omega^n$ are cohomologous. (Note that $\mathcal{L}_X \beta = 0$ because $\psi_t^* \beta = \beta$.)

On the other hand, $\iota(X) \sigma$ is Poincaré dual to $\alpha$, because if we let $D$ be a fiber of the normal bundle of $\gamma$ in $M$, then

$$\int_D 2\pi \cdot \iota(X) \sigma = \frac{1}{2\pi} \int_{\gamma \times D} 2\pi \cdot \sigma = \int_M \sigma = 1.$$

The lemma follows immediately. \hfill \Box

A compact, closed symplectic manifold $(M, \omega)$ of dimension $2n$ is said of Lefschetz type if

$$\wedge^{n-1} : H^1(M; \mathbb{R}) \rightarrow H^{2n-1}(M; \mathbb{R}) : \alpha \mapsto \alpha \cup [\omega]^{n-1}, \forall \alpha \in H^1(M; \mathbb{R})$$

is an isomorphism. It is known that compact Kähler manifolds are of Lefschetz type.

**Proposition 1.14.** Suppose $(M, \omega)$ is a compact, closed symplectic manifold of Lefschetz type. Then a symplectic $S^1$-action on $(M, \omega)$ is Hamiltonian if and only if there is a fixed point.

**Proof.** It suffices to show that the $S^1$-action is Hamiltonian if it has a fixed point. Suppose the action has a fixed point. Then the homology class of an orbit must be zero because it shrinks to a fixed point. This implies by Lemma 1.13 that its Poincaré dual $2\pi \cdot [\iota(X) \omega^n]$, which is the image of $2\pi \cdot [\iota(X) \omega]$ under $\wedge^{n-1} : H^1(M; \mathbb{R}) \rightarrow H^{2n-1}(M; \mathbb{R})$, is also zero. But $(M, \omega)$ is of Lefschetz type so that $\wedge^{n-1}$ is isomorphic. This shows that $\iota(X) \omega$ is exact, and the $S^1$-action is Hamiltonian. \hfill \Box

Let $(M, \omega)$ be a compact, closed symplectic manifold. If $\dim M \leq 4$, then every symplectic $S^1$-action which has a fixed point is Hamiltonian. McDuff found the first example of symplectic $S^1$-actions on a 6-dimensional manifold which has a fixed point but is not Hamiltonian. In her example the fixed points are not isolated. One has not been able to find an example which has only isolated fixed points but is not Hamiltonian.
Conjecture 1.15. Let $(M, \omega)$ be a 6-dimensional compact, closed symplectic manifold. Then a symplectic $\mathbb{S}^1$-action on $(M, \omega)$ is Hamiltonian if there is a fixed point and all fixed points are isolated.

2. Hamiltonian torus actions

We denote by $\mathbb{T}^n = (\mathbb{S}^1)^n$ the $n$-torus. The corresponding Lie algebra and its dual are denoted by $\mathfrak{t}^n$ and $(\mathfrak{t}^n)^*$ respectively. Since $\mathbb{T}^n$ is abelian, the Lie bracket is trivial, and $\mathfrak{t}^n$ and $(\mathfrak{t}^n)^*$ can be canonically identified with $\mathbb{R}^n$, with the pairing between $\mathfrak{t}^n$ and $(\mathfrak{t}^n)^*$ given by the inner product on $\mathbb{R}^n$.

Let $(M, \omega)$ be a symplectic manifold, and let $\mathbb{T}^n$ act (effectively) on $M$ via symplectomorphisms. Then for any $\xi \in \mathfrak{t}^n$, one has a 1-parameter group of symplectomorphisms $\exp(t\xi)$. We denote by $X_\xi$ the vector field on $M$ which generates the flow $\exp(t\xi)$. Note that for any $\xi, \eta \in \mathfrak{t}^n$, $[X_\xi, X_\eta] = X_{[\xi, \eta]} = 0$ since $\mathbb{T}^n$ is abelian. On the other hand, each $X_\xi$ is a symplectic vector field, i.e., $\iota(X_\xi)\omega$ is closed. We say that a symplectic $\mathbb{T}^n$-action on $(M, \omega)$ is weakly Hamiltonian if for any $\xi \in \mathfrak{t}^n$, $\iota(X_\xi)\omega = dH_\xi$ for some smooth function $H_\xi$ on $M$ (note that $H_\xi$ is uniquely determined up to a constant).

In order to define Hamiltonian actions, we recall the concept of Poisson bracket. Let $F, H$ be smooth functions on $M$. We denote by $X_F, X_H$ the corresponding Hamiltonian vector fields, i.e., $\iota(X_F)\omega = dF, \iota(X_H)\omega = dH$. Then the Poisson bracket of $F, H$ is defined and denoted by

$$\{F, H\} \equiv \omega(X_F, X_H) = dF(X_H) = -dH(X_F).$$

In particular, $\{F, H\} = 0$ means that the Hamiltonian function $F$ is constant under the flow generated by $X_H$ (and vice versa). The set of smooth functions on $(M, \omega)$ becomes a Lie algebra under the Poisson bracket.

With the above understood, a weakly Hamiltonian $\mathbb{T}^n$-action is called Hamiltonian if for any $\xi, \eta \in \mathfrak{t}^n$, the Poisson bracket $\{H_\xi, H_\eta\} = 0$. (In general, a weakly Hamiltonian Lie group action is called Hamiltonian if $\xi \mapsto H_\xi$ can be chosen to be a Lie algebra homomorphism.)

The next lemma shows that in many cases a weakly Hamiltonian $\mathbb{T}^n$-action is automatically Hamiltonian.

Lemma 2.1. For any weakly Hamiltonian $\mathbb{T}^n$-action on $(M, \omega)$, the Poisson bracket $\{H_\xi, H_\eta\}$ is a constant function on $M$ for any $\xi, \eta \in \mathfrak{t}^n$. In particular, a weakly Hamiltonian $\mathbb{T}^n$-action is Hamiltonian if $\exp(t\xi)$ has a fixed point for any $\xi \in \mathfrak{t}^n$ (e.g., when $M$ is compact, closed).

Proof. It follows from the following straightforward calculation.
Here we use the fact that \( \iota(X_{\xi})\omega, \iota(X_{\eta})\omega \) are closed, and the assumption that \( \mathbb{T}^n \) is abelian so that \([X_{\eta}, X_{\xi}] = 0\).

\[\]

The **moment map** of a Hamiltonian \( \mathbb{T}^n \)-action on \((M, \omega)\) is a smooth map

\[
m: M \to (\mathfrak{t}^n)^* = \mathbb{R}^n,
\]

such that for any \( \xi \in \mathfrak{t}^n = \mathbb{R}^n \),

\[
H_{\xi}(p) = \langle \mu(p), \xi \rangle, \forall p \in M,
\]

is a Hamiltonian function for \( \exp(t\xi) \), i.e., \( \iota(X_{\xi})\omega = dH_{\xi} \). Note that the assignment \( \xi \mapsto H_{\xi} \) is linear.

**Remark 2.2.** (1) The moment map always exists. For example, let \( \xi_1, \ldots, \xi_n \in \mathfrak{t}^n \) be a basis, and let \( \xi_1^*, \ldots, \xi_n^* \in (\mathfrak{t}^n)^* \) be the corresponding dual basis. Then

\[
\mu(p) = H_{\xi_1}(p)\xi_1^* + \cdots + H_{\xi_n}(p)\xi_n^*, \forall p \in M,
\]

is a moment map.

(2) The moment map is uniquely defined up to a constant vector in \((\mathfrak{t}^n)^*\).

(3) Because of the condition \( \{H_{\xi}, H_{\eta}\} = 0 \) for any \( \xi, \eta \in \mathfrak{t}^n \) and the fact that \( \mathbb{T}^n \) is connected, the moment map \( \mu: M \to \mathbb{R}^n \) is \( \mathbb{T}^n \)-invariant, i.e., \( \mu(g \cdot p) = \mu(p) \) for any \( g \in \mathbb{T}^n \).

Let \( p \) be a point in \( M \). We next give a description of the image of \( d\mu_p : T_p M \to (\mathfrak{t}^n)^* = \mathbb{R}^n \). Let us consider the subspace of \( \mathfrak{t}^n = \mathbb{R}^n \) which annihilates the image, i.e., the set of \( \xi \in \mathfrak{t}^n = \mathbb{R}^n \) such that \( \langle d\mu_p(Y), \xi \rangle = 0 \) for all \( Y \in T_p M \). Observe the identity \( \langle d\mu_p(Y), \xi \rangle = \langle dH_{\xi} \rangle_p(Y) = \omega_p(X_{\xi}, Y) \). Since \( \omega \) is nondegenerate, we see immediately that the set of \( \xi \) which annihilates the image of \( d\mu_p : T_p M \to (\mathfrak{t}^n)^* = \mathbb{R}^n \) is the subspace \( \{\xi \in \mathfrak{t}^n | X_{\xi} = 0 \text{ at } p\} \), or equivalently, the subspace \( \{\xi \in \mathfrak{t}^n | p \text{ is a fixed point of the subgroup } \exp(t\xi)\} \). In particular, since the principal orbit, i.e., the set of points in \( M \) which has trivial isotropy, is open and dense for an effective action, we see that the set of regular values of the moment map \( \mu : M \to (\mathfrak{t}^n)^* = \mathbb{R}^n \) is open and dense in the image \( \mu(M) \).

Let \( \lambda \in (\mathfrak{t}^n)^* = \mathbb{R}^n \) be a regular value of \( \mu \). Since \( \mu \) is \( \mathbb{T}^n \)-invariant, we see that the level surface \( \mu^{-1}(\lambda) \) is \( \mathbb{T}^n \)-invariant. The quotient space \( B_\lambda \equiv \mu^{-1}(\lambda)/\mathbb{T}^n \), which is an orbifold in general of dimension \( \dim M - 2n \), has a natural symplectic structure \( \omega_\lambda \). The space \((B_\lambda, \omega_\lambda)\) is called the reduced space at \( \lambda \) (its proof is similar to the case of
S\(^1\)-action, cf. Proposition 1.1). Note that \(\dim M - 2n \geq 0\), namely, the dimension of the torus is at most half of the dimension of the symplectic manifold which the torus acts on. When the dimension of the torus equals half of the dimension of the symplectic manifold, the reduced spaces consist of single points, and the preimages \(\mu^{-1}(\lambda)\) are orbits of the torus action, which are easily seen to be embedded Lagrangian tori (they are Lagrangian because of the condition \(\{H_\xi, H_\eta\} = \omega(X_\xi, X_\eta) = 0\) for any \(\xi, \eta \in t^n\).

The fundamental result concerning Hamiltonian torus actions is the following convexity theorem, due to Atiyah and Guillemin-Sternberg independently.

**Theorem 2.3. (Atiyah-Guillemin-Sternberg).** Let \((M, \omega)\) be a compact, connected symplectic manifold which is equipped with a Hamiltonian \(\mathbb{T}^n\)-action of moment map \(\mu : M \to \mathbb{R}^n\). Then the fixed points of the \(\mathbb{T}^n\)-action form a finite union of connected symplectic submanifolds \(Q_1, \ldots, Q_N\), such that on each \(Q_j\), the moment map \(\mu\) has a constant value \(\lambda_j \in \mathbb{R}\), and the image of \(\mu\) is the convex hull of \(\lambda_j\), i.e.,

\[
\mu(M) = \left\{ \sum_{j=1}^{N} x_j \lambda_j \mid \sum_{j=1}^{N} x_j = 1, x_j \geq 0 \right\} \subset \mathbb{R}^n.
\]

**Example 2.4.** (1) \(\mathbb{T}^n\)-action on \(\mathbb{CP}^n\). Consider the following Hamiltonian \(\mathbb{T}^n\)-action on \(\mathbb{CP}^n\)

\[
(t_1, t_2, \ldots, t_n) \cdot [z_0, z_1, z_2, \ldots, z_n] = [z_0, e^{-it_1} z_1, e^{-it_2} z_2, \ldots, e^{-it_n} z_n],
\]

which has moment map

\[
\mu([z_0, z_1, \cdots, z_n]) = \frac{1}{2} \left( \frac{|z_1|^2}{|z_0|^2 + \cdots + |z_n|^2}, \cdots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2} \right).
\]

Clearly \(\mu(\mathbb{CP}^n) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq \frac{1}{2}, x_i \geq 0\}\). The fixed points are \(p_0 = [1, 0, \cdots, 0], p_1 = [0, 1, \cdots, 0], \cdots, p_n = [0, 0, \cdots, 1]\), which are mapped under \(\mu\) to \(\lambda_0 = (0, 0, \cdots, 0), \lambda_1 = (\frac{1}{2}, 0, \cdots, 0), \cdots, \lambda_n = (0, 0, \cdots, \frac{1}{2})\) respectively. \(\mu(\mathbb{CP}^n)\) is the \(n\)-simplex with vertices \(\lambda_0, \lambda_1, \cdots, \lambda_n\).

(2) A non-effective \(\mathbb{T}^2\)-action on \(\mathbb{CP}^2\). Consider the following non-effective action

\[
(t_1, t_2) \cdot [z_0, z_1, z_2] = [z_0, e^{-it_1} z_1, e^{-2it_2} z_2],
\]

which has moment map

\[
\mu([z_0, z_1, z_2]) = \frac{1}{2} \left( \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{2|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right).
\]

The image \(\mu(\mathbb{CP}^2)\) is the triangle with vertices \((0, 0), (\frac{1}{2}, 0)\) and \((0, 1)\).

(3) \(\mathbb{T}^2\)-actions on \(\mathbb{CP}^1 \times \mathbb{CP}^1\). Consider the following action

\[
(t_1, t_2) \cdot ([z_0, z_1], [w_0, w_1]) = ([z_0, e^{-it_1} z_1], [w_0, e^{-it_2} w_1]).
\]

The moment map is

\[
\mu([z_0, z_1], [w_0, w_1]) = \frac{1}{2} \left( \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}, \frac{|w_0|^2}{|w_0|^2 + |w_1|^2} \right).
\]
and the fixed points are \([(1,0), [1,0]), ([1,0], [0,1]), ([0,1], [1,0])\), with values under \(\mu\) being \((0,0), (0,\frac{1}{2}), (\frac{1}{2},0)\) and \((\frac{1}{2},\frac{1}{2})\) respectively. The image of the moment map is

\[
\mu(\mathbb{CP}^1 \times \mathbb{CP}^1) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1, x_2 \leq \frac{1}{2}\}.
\]

Consider the following (effective) \(T^2\)-action on \(\mathbb{CP}^1 \times \mathbb{CP}^1\):

\[
(t_1, t_2) \cdot ([z_0, z_1], [w_0, w_1]) = ([z_0, e^{-it_1} z_1], [w_0, e^{-it_1 - it_2} w_1]).
\]

The moment map is

\[
\mu([z_0, z_1], [w_0, w_1]) = \frac{1}{2} \left( \frac{|z_1|^2}{|z_0|^2 + |z_1|^2} + \frac{|w_0|^2}{|w_0|^2 + |w_1|^2} \right),
\]

and the fixed points are \([(1,0), [1,0]), ([1,0], [0,1]), ([0,1], [1,0])\), with values under \(\mu\) being \((0,0), (\frac{3}{2},\frac{1}{2}), (\frac{3}{2},\frac{1}{2})\) respectively. The image of \(\mu\) is the parallelogram with vertices \((0,0), (\frac{1}{2},\frac{1}{2}), (\frac{3}{4},\frac{1}{4})\) and \((\frac{3}{2},\frac{1}{2})\).

Consider the following (effective) \(T^2\)-action on \(\mathbb{CP}^1 \times \mathbb{CP}^1\):

\[
(t_1, t_2) \cdot ([z_0, z_1], [w_0, w_1]) = ([z_0, e^{-it_1} z_1], [w_0, e^{-2it_1 - it_2} w_1]).
\]

The moment map is

\[
\mu([z_0, z_1], [w_0, w_1]) = \frac{1}{2} \left( \frac{|z_1|^2}{|z_0|^2 + |z_1|^2} + \frac{|2w_0|^2}{|w_0|^2 + |w_1|^2} \right),
\]

and the fixed points are \([(1,0), [1,0]), ([1,0], [0,1]), ([0,1], [1,0])\), with values under \(\mu\) being \((0,0), (1,\frac{1}{2}), (\frac{1}{2},0)\) and \((\frac{3}{4},\frac{1}{4})\) respectively. The image of \(\mu\) is the parallelogram with vertices \((0,0), (1,\frac{1}{2}), (\frac{1}{2},0)\) and \((\frac{3}{4},\frac{1}{4})\).

(4) A \(T^2\)-action on Hirzebruch surface \(\mathbb{CP}^2 \# \mathbb{CP}^2\). Here the Hirzebruch surface is given as the complex surface

\[
M = \{([a,b], [x,y,z]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 \mid ay = bx\}.
\]

The \(T^2\)-action on \(M\) is the restriction of the following \(T^2\)-action on \(\mathbb{CP}^1 \times \mathbb{CP}^2\):

\[
(t_1, t_2) \cdot ([a,b], [x,y,z]) = ([e^{-it_1}a,b], [e^{-it_1}x,y,e^{-it_2}z]),
\]

which leaves \(M\) invariant.

The moment map is

\[
\mu([a,b], [x,y,z]) = \frac{1}{2} \left( \frac{|a|^2}{|a|^2 + |b|^2} + \frac{|x|^2}{|x|^2 + |y|^2 + |z|^2} \right),
\]

and there are four fixed points on \(M\), which are

\[
\left\{([1,0], [1,0]), ([1,0], [0,0,1]), ([0,1], [1,0]), ([0,1], [0,0,1])\right\}.
\]

The corresponding values under the moment map are \((1,0), (\frac{1}{2},\frac{1}{2}), (0,0)\) and \((0,\frac{1}{2})\).

Let \((M, \omega)\) be a symplectic manifold with a Hamiltonian \(T^n\)-action and let \(\mu : M \to (t^n)^*\) be the corresponding moment map. For any \(k < n\) let \(T^k \subset T^n\) be a sub-torus. Then there is naturally an induced Hamiltonian \(T^k\)-action on \(M\). The moment map of the induced action is \(\mu : M \to (t^n)^*\) composed with the projection \((t^n)^* \to (t^k)^*\).
Example 2.5. Consider the $S^1$-action on $\mathbb{CP}^2$ which is induced from the standard $T^2$-action on $\mathbb{CP}^2$ considered in Example 2.4 (1) by the embedding $S^1 \subset T^2$ given by $t \mapsto (t, 2t)$. This is the same $S^1$-action we considered in Example 1.10 (3).

Note that the image of the moment map of the $T^2$-action on $\mathbb{CP}^2$ is the triangle with vertices $(0, 0)$, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Its projection onto the line $\mathbb{R}\langle 1, 2 \rangle \subset \mathbb{R}^2$ is the line segment between the points $(0, 0)$ and $(\frac{1}{5}, \frac{2}{5})$, which are the images of the vertices $(0, 0)$ and $(0, \frac{1}{2})$ under the projection. Note that the image of the vertex $(\frac{1}{2}, 0)$ is the middle point $(\frac{1}{10}, \frac{1}{5})$ of the line segment. Compare with the moment map in Example 1.10 (3) and notice that the length of the vector $\langle 1, 2 \rangle$ is $\sqrt{5}$.

In general the set of regular values of the moment map is divided into several chambers. We illustrate this with the following example of a $T^2$-action on $\mathbb{CP}^3$.

Example 2.6. Consider the Hamiltonian $T^2$-action on $\mathbb{CP}^3$

$$(t_1, t_2) \cdot [z_0, z_1, z_2, z_3] = [z_0, e^{-it_1}z_1, e^{-2it_1}z_2, e^{-it_2}z_3],$$

which has moment map

$$\mu([z_0, z_1, z_2, z_3]) = \frac{1}{2}(\frac{|z_1|^2 + 2|z_2|^2}{\sum_{j=0}^{3}|z_j|^2}, \frac{|z_3|^2}{\sum_{j=0}^{3}|z_j|^2}).$$

The image of $\mu$ is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, \frac{1}{2})$. The set of regular values of $\mu$ is the interior of the triangle with the line segment between the points $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ removed. So it is divided into two chambers by the line segment. Notice that the “wall” that divides the two chambers is the image of $\{(0, z, w) \in \mathbb{CP}^3\}$ under $\mu$, which is fixed by the diagonal sub-torus $\{(t, t)\} \subset T^2$.

A compact, connected symplectic manifold of dimension $2n$ is called toric if it admits an effective Hamiltonian $T^n$-action. Delzant showed that such a space together with the $T^n$-action is uniquely determined by the image of the moment map, and moreover, there exists a $T^n$-invariant complex structure with respect to which the symplectic form is Kähler.

Compact, connected symplectic 4-manifolds with a Hamiltonian $S^1$-action have been classified, from which it is known that such spaces are all Kähler and the $S^1$-actions are holomorphic. However, S. Tolman constructed an example of a Hamiltonian $T^2$-action on a compact, connected 6-dimensional manifold which does not admit any $S^1$-invariant holomorphic Kähler structure.

References