LECTURE 2: SYMPLECTIC VECTOR BUNDLES

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1. Symplectic Vector Spaces

**Definition 1.1.** A symplectic vector space is a pair \((V,\omega)\) where \(V\) is a finite dimensional vector space (over \(\mathbb{R}\)) and \(\omega\) is a bilinear form which satisfies

- Skew-symmetry: for any \(u, v \in V\),
  \[\omega(u, v) = -\omega(v, u)\].
- Nondegeneracy: for any \(u \in V\),
  \[\omega(u, v) = 0 \forall v \in V\] implies \(u = 0\).

A linear symplectomorphism of a symplectic vector space \((V,\omega)\) is a vector space isomorphism \(\psi: V \to V\) such that
\[\omega(\psi u, \psi v) = \omega(u, v) \forall u, v \in V.\]
The group of linear symplectomorphisms of \((V,\omega)\) is denoted by \(\text{Sp}(V,\omega)\).

**Example 1.2.** The Euclidean space \(\mathbb{R}^{2n}\) carries a standard skew-symmetric, nondegenerate bilinear form \(\omega_0\) defined as follows. For \(u = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T,\)
\[u' = (x_1', x_2', \cdots, x_n', y_1', y_2', \cdots, y_n')^T,\]
\[\omega_0(u, u') = \sum_{i=1}^{n}(x_i y'_i - x'_i y_i) = -u^T J_0 u',\]
where \(J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\). (Here \(I\) is the \(n \times n\) identity matrix.)

The group of linear symplectomorphisms of \((\mathbb{R}^{2n},\omega_0)\), which is denoted by \(\text{Sp}(2n)\), can be identified with the group of \(2n \times 2n\) symplectic matrices. Recall a symplectic matrix \(\Psi\) is one which satisfies \(\Psi^T J_0 \Psi = J_0\). For the case of \(n = 1\), a symplectic matrix is simply a matrix \(\Psi\) with \(\det \Psi = 1\).

Let \((V,\omega)\) be any symplectic vector space, and let \(W \subset V\) be any linear subspace. The **symplectic complement** of \(W\) in \(V\) is defined and denoted by
\[W^\omega = \{v \in V | \omega(v, w) = 0, \forall w \in W\}.\]

**Lemma 1.3.** (1) \(\dim W + \dim W^\omega = \dim V\), (2) \((W^\omega)^\omega = W\).

*Proof.* Define \(\iota\omega: V \to V^*\) by \(\iota\omega(v): w \mapsto \omega(v, w), \forall v, w \in V\), where \(V^*\) is the dual space of \(V\). Since \(\omega\) is nondegenerate, \(\iota\omega\) is an isomorphism. Now observe that \(\iota\omega(W^\omega) = W^\perp\) where \(W^\perp \subset V^*\) is the annihilator of \(W\), i.e.,
\[W^\perp = \{v^* \in V^* | v^*(w) = 0, \forall w \in W\}.\]
Part (1) follows immediately from the fact that
\[ \dim W + \dim W^\perp = \dim V. \]

Part (2) follows easily from \( W \subset (W^\omega)^\omega \) and the equations
\[ \dim W = \dim V - \dim W^\omega = \dim(W^\omega)^\omega \]
which are derived from (1).

**Theorem 1.4.** For any symplectic vector space \((V, \omega)\), there exists a basis of \( V \)
\( u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n \) such that
\[ \omega(u_j, u_k) = \omega(v_j, v_k) = 0, \quad \omega(u_j, v_k) = \delta_{jk}. \]
(Such a basis is called a symplectic basis.) In particular, \( \dim V = 2n \) is even.

**Proof.** We prove by induction on \( \dim V \). Note that \( \dim V \geq 2 \).

When \( \dim V = 2 \), the nondegeneracy condition of \( \omega \) implies that there exist \( u, v \in V \)
such that \( \omega(u, v) \neq 0 \). Clearly \( u, v \) are linearly independent so that they form a basis
of \( V \) since \( \dim V = 2 \). We can replace \( v \) by an appropriate nonzero multiple so that the
condition \( \omega(u, v) = 1 \) is satisfied. Hence the theorem is true for the case of \( \dim V = 2 \).

Now suppose the theorem is true when \( \dim V \leq m - 1 \). We shall prove that it is
also true when \( \dim V = m \). Again the nondegeneracy condition of \( \omega \) implies that
there exist \( u_1, v_1 \in V \) such that \( u_1, v_1 \) are linearly independent and \( \omega(u_1, v_1) = 1 \). Set
\( W \equiv \text{span}(u_1, v_1) \). Then we claim that \((W^\omega, \omega|_{W^\omega})\) is a symplectic vector space. It
suffices to show that \( \omega|_{W^\omega} \) is nondegenerate. To see this, suppose \( w \in W^\omega \) such that
\( \omega(w, z) = 0 \) for all \( z \in W^\omega \). We need to show that \( w \) must be zero. To this end, note
that \( W \cap W^\omega = \{0\} \), so that \( V = W \oplus W^\omega \) by (1) of the previous lemma. Now for
any \( z \in V \), write \( z = z_1 + z_2 \) where \( z_1 \in W \) and \( z_2 \in W^\omega \). Then \( \omega(w, z_1) = 0 \) because
\( w \in W^\omega \) and \( \omega(w, z_2) = 0 \) by \( z_2 \in W^\omega \) and the assumption on \( w \). Hence \( \omega(w, z) = 0 \),
and therefore \( w = 0 \) by the nondegeneracy condition of \( \omega \) on \( V \).

Note that \( \dim W^\omega = \dim V - 2 \leq m - 1 \), so that by the induction hypothesis,
there is a symplectic basis \( u_2, \cdots, u_n, v_2, \cdots, v_n \) of \((W^\omega, \omega|_{W^\omega})\). It is clear that
\( u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n \) is a symplectic basis of \((V, \omega)\), and the theorem is proved. \(\square\)

**Corollary 1.5.** Let \( \omega \) be any skew-symmetric bilinear form on \( V \). Then \( \omega \) is nondegenerate if and only if \( \dim V = 2n \) is even and
\[ \omega^n = \omega \wedge \cdots \wedge \omega \neq 0. \]

**Proof.** Suppose \( \omega \) is nondegenerate, then by the previous theorem, \( \dim V = 2n \) is even,
and there exists a symplectic basis \( u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n \). Clearly
\[ \omega^n(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) \neq 0, \]
hence \( \omega^n \neq 0 \).

Suppose \( \dim V = 2n \) is even and \( \omega^n \neq 0 \). Then \( \omega \) must be nondegenerate, because
if otherwise, there exists a \( u \in V \) such that \( \omega(u, v) = 0 \) for all \( v \in V \). We complete \( u \)
into a basis \( u, v_1, v_2, \cdots, v_{2n-1} \) of \( V \). One can easily check that
\[ \omega^n(u, v_1, v_2, \cdots, v_{2n-1}) = 0, \]
which contradicts the assumption that \( \omega^n \neq 0 \).

\[ \square \]

**Corollary 1.6.** Let \((V, \omega)\) be any symplectic vector space. Then there exists an \( n > 0 \) and a vector space isomorphism \( \phi : \mathbb{R}^{2n} \to V \) such that

\[
\omega_{0}(z, z') = \omega(\phi z, \phi z'), \forall z, z' \in \mathbb{R}^{2n}.
\]

Consequently, \( \text{Sp}(V, \omega) \) is isomorphic to \( \text{Sp}(2n) \).

**Proof.** Let \( u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n \) be a symplectic basis of \((V, \omega)\). The corollary follows by defining

\[
\phi : (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto \sum_{i=1}^{n} (x_i u_i + y_i v_i).
\]

\[ \square \]

**Definition 1.7.** A complex structure on a real vector space \( V \) is an automorphism \( J : V \to V \) such that \( J^2 = -\text{id} \). A Hermitian structure on \((V, J)\) is an inner product \( g \) on \( V \) which is \( J \)-invariant, i.e., \( g(Jv, Jw) = g(v, w) \), for all \( v, w \in V \).

Let \( J \) be a complex structure on \( V \). Then \( V \) becomes a complex vector space by defining the complex multiplication by

\[
\mathbb{C} \times V \to V : (x + iy, v) \mapsto xv + yJv.
\]

If \( g \) is a Hermitian structure on \((V, J)\), then there is an associated Hermitian inner product

\[
h(v, w) \equiv g(v, w) + ig(v, Jw), \forall v, w \in V,
\]

i.e., \( h : V \times V \to \mathbb{C} \) satisfies (1) \( h \) is complex linear in the first \( V \) and anti-complex linear in the second \( V \), and (2) \( h(v, v) > 0 \) for any \( 0 \neq v \in V \).

Note that there always exists a Hermitian structure on \((V, J)\), by simply taking the average \( \frac{1}{2}(g(v, w) + g(Jv, Jw)) \) of any inner product \( g \) on \( V \).

**Definition 1.8.** Let \((V, \omega)\) be a symplectic vector space. A complex structure \( J \) on \( V \) is called \( \omega \)-compatible if

- \( \omega(Jv, Jw) = \omega(v, w) \) for all \( v, w \in V \),
- \( \omega(v, Jv) > 0 \) for any \( 0 \neq v \in V \).

We denote the set of \( \omega \)-compatible complex structures by \( \mathcal{J}(V, \omega) \). Note that \( \mathcal{J}(V, \omega) \) is nonempty: let \( u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n \) be a symplectic basis of \((V, \omega)\), then \( J : V \to V \) defined by \( Ju_i = v_i, Jv_i = -u_i \) is a \( \omega \)-compatible complex structure. Finally, for any \( J \in \mathcal{J}(V, \omega) \), \( g_J(v, w) \equiv \omega(v, Jw), \forall v, w \in V \), is a (canonically associated) Hermitian structure on \((V, J)\).

**Example 1.9.** \( J_0 \) is a complex structure on \( \mathbb{R}^{2n} \) which is \( \omega_0 \)-compatible. The associated Hermitian structure \( g_0(\cdot, \cdot) \equiv \omega_0(\cdot, J_0 \cdot) \) is the usual inner product on \( \mathbb{R}^{2n} \). \( J_0 \) makes \( \mathbb{R}^{2n} \) into a complex vector space by

\[
\mathbb{C} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} : (x + iy, v) \mapsto xv + yJ_0v.
\]
which coincides with the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T \mapsto (z_1, z_2, \cdots, z_n)^T$$

where $z_j = x_j + iy_j$ with $i = \sqrt{-1}$. With this understood, the associated Hermitian inner product $h_0 \equiv g_0 - i\omega_0$ on $(\mathbb{R}^{2n}, J_0)$ is the usual one on $\mathbb{C}^n$: $h_0(z, w) = w^T z$.

We would like to understand the set $J(V, \omega)$.

**Lemma 1.10.** Suppose $\dim V = 2n$. Then for any $J \in J(V, \omega)$, there is a vector space isomorphism $\phi_J : \mathbb{R}^{2n} \to V$ such that

$$\phi_J^* \omega = \omega_0, \quad \phi_J^* J \equiv \phi_J^{-1} \circ J \circ \phi_J = J_0.$$ 

Moreover, $\phi_J : J' \mapsto \phi_J^{-1} \circ J' \circ \phi_J$ identifies $J(V, \omega)$ with $J(\mathbb{R}^{2n}, \omega_0)$.

**Proof.** Let $u_1, u_2, \cdots, u_n$ be a unitary basis of $(V, h_J)$, where $h_J$ is the Hermitian inner product on $V$ associated to the canonical Hermitian structure $g_J(\cdot, \cdot) \equiv \omega(\cdot, J \cdot)$. Then one can easily check that $u_1, u_2, \cdots, u_n, J u_1, J u_2, \cdots, J u_n$ form a symplectic basis of $(V, \omega)$. If we define

$$\phi_J : (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) \mapsto \sum_{i=1}^{n} (x_i u_i + y_i J u_i),$$

then it is obvious that $\phi_J^* \omega = \omega_0$ and $\phi_J^* J = J_0$. Moreover, the verification that $\phi_J : J' \mapsto \phi_J^{-1} \circ J' \circ \phi_J$ identifies $J(V, \omega)$ with $J(\mathbb{R}^{2n}, \omega_0)$ is straightforward. $\square$

We remark that in the proof the definition of $\phi_J$ may depend on the choice of the unitary basis $u_1, u_2, \cdots, u_n$, but the identification $\phi_J^* : J(V, \omega) \to J(\mathbb{R}^{2n}, \omega_0)$ does not, it is completely determined by $J$.

In order to understand $J(\mathbb{R}^{2n}, \omega_0)$, we need to go over some basic facts about the Lie group $Sp(2n)$. To this end, recall that $\mathbb{R}^{2n}$ is identified with $\mathbb{C}^n$ by

$$(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T \mapsto (x_1, x_2, \cdots, z_n)^T,$$

where $z_j = x_j + iy_j$.

Under this identification, $GL(n, \mathbb{C})$ is regarded as a subgroup of $GL(2n, \mathbb{R})$ via

$$Z = X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}. $$

Note that $\psi \in GL(2n, \mathbb{R})$ belongs to $GL(n, \mathbb{C})$ iff $\psi J_0 = J_0 \psi$.

**Lemma 1.11.**

$$U(n) = Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n, \mathbb{C}) = O(2n) \cap GL(n, \mathbb{C}).$$

**Proof.** Let $\psi \in GL(2n, \mathbb{R})$, then

- $\psi \in GL(n, \mathbb{C})$ iff $\psi J_0 = J_0 \psi$,
- $\psi \in Sp(2n)$ iff $\psi^T J_0 \psi = J_0$,
- $\psi \in O(2n)$ iff $\psi^T \psi = I$. 

The last two identities in the lemma follows immediately.

It remains to show that a $\psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in GL(n, \mathbb{C})$ lies in $\text{Sp}(2n)$ iff $\psi = X + iY$ lies in $U(n)$. But one can check easily that both conditions are equivalent to the following set of equations

$$X^T Y = Y^T X, \quad X^T X + Y^T Y = I.$$  

The lemma follows immediately.  

The previous lemma in particular says that $U(n)$ is a subgroup of $\text{Sp}(2n)$. The space of right orbits

$$\text{Sp}(2n)/U(n) \equiv \{ \psi \cdot U(n) | \psi \in \text{Sp}(2n) \}$$

is known to be naturally a smooth manifold. We denote by $\ast$ the orbit of identity $I \in \text{Sp}(2n)$ in $\text{Sp}(2n)/U(n)$.

**Theorem 1.12.** There exists a canonically defined smooth map

$$H : \text{Sp}(2n)/U(n) \times [0, 1] \rightarrow \text{Sp}(2n)/U(n)$$

such that $H(\ast, 0) = \text{id}$, $H(\ast, 1)(x) = \{ \ast \}$ for any $x \in \text{Sp}(2n)/U(n)$, and $H(\ast, t) = \ast$ for any $t \in [0, 1]$. In particular, $\text{Sp}(2n)/U(n)$ is contractible.

**Proof.** First of all, for any $\psi \in \text{Sp}(2n)$, $\psi^T$ is also in $\text{Sp}(2n)$, so that $\psi \psi^T$ is a symmetric, positive definite symplectic matrix. We will show that $(\psi \psi^T)^\alpha$ is also a symplectic matrix for any real number $\alpha$.

To this end, we decompose $\mathbb{R}^{2n} = \bigoplus \lambda V_\lambda$ where $V_\lambda$ is the $\lambda$-eigenspace of $\psi \psi^T$, and $\lambda > 0$. Then note that for any $z \in V_\lambda$, $z' \in V_{\lambda'}$, $\omega_0(z, z') = 0$ if $\lambda \lambda' \neq 1$. Our claim that $(\psi \psi^T)^\alpha$ is a symplectic matrix for any real number $\alpha$ follows easily from this observation.

Now for any $\psi \in \text{Sp}(2n)$, we decompose $\psi = PQ$ where $P = (\psi \psi^T)^{1/2}$ is symmetric and $Q \in O(2n)$. Note that $Q = \psi P^{-1} \in \text{Sp}(2n) \cap O(2n) = U(n)$, which shows that $\psi$ and $P = (\psi \psi^T)^{1/2}$ are in the same orbit in $\text{Sp}(2n)/U(n)$. With this understood, we define

$$H : (\psi \cdot U(n), t) \mapsto (\psi \psi^T)^{(1-t)/2} \cdot U(n), \quad \psi \in \text{Sp}(2n), \ t \in [0, 1].$$

**Lemma 1.13.** $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ is canonically identified with $\text{Sp}(2n)/U(n)$, under which $J_0$ is sent to $\ast$.

**Proof.** By Lemma 1.11, for any $J \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$, there exists a $\phi_J \in \text{Sp}(2n)$ such that $\phi_J^* J \equiv \phi_J^{-1} \cdot J \cdot \phi_J = J_0$, or equivalently, $J = \phi_J \cdot J_0 \cdot \phi_J^{-1}$. The correspondence $J \mapsto \phi_J$ induces a map from $\mathcal{J}(\mathbb{R}^{2n}, \omega_0)$ to $\text{Sp}(2n)/U(n)$, which is clearly one to one and onto.

Note that under the correspondence, $J_0$ is sent to $\ast$.

The set of $\omega$-compatible complex structures $\mathcal{J}(V, \omega)$ can be given a natural topology so that it becomes a smooth manifold. Lemma 1.10, Theorem 1.12 and Lemma 1.13 give rise to the following
Corollary 1.14. Given any $J \in \mathcal{J}(V, \omega)$, there exists a canonically defined smooth map

$$H_J : \mathcal{J}(V, \omega) \times [0,1] \to \mathcal{J}(V, \omega)$$

depending smoothly on $J$, such that $H_J(\cdot, 0) = \text{id}$, $H_J(\cdot, 1)(J') = \{J\}$ for any $J' \in \mathcal{J}(V, \omega)$, and $H(J, t) = J$ for any $t \in [0,1]$.

Recall that for any $J \in \mathcal{J}(V, \omega)$, there is a canonically associated Hermitian structure (i.e. a $J$-invariant inner product) $g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$. The next theorem shows that one can construct $\omega$-compatible complex structures from inner products on $V$. Let $\text{Met}(V)$ denote the space of inner products on $V$.

Theorem 1.15. There exists a canonically defined map $r : \text{Met}(V) \to \mathcal{J}(V, \omega)$ such that

$$r(g_J) = J, \quad r(\psi^* g) = \psi^* r(g)$$

for all $J \in \mathcal{J}(V, \omega)$, $g \in \text{Met}(V)$, and $\psi \in \text{Sp}(V, \omega)$.

Proof. For any given $g \in \text{Met}(V)$, we define $A : V \to V$ by

$$\omega(v, w) = g(Av, w), \quad \forall v, w \in V.$$ 

Then the skew-symmetry of $\omega$ implies that $A$ is $g$-skew-adjoint. It follows that $P \equiv -A^2$ is $g$-self-adjoint and $g$-positive definite. Set $Q \equiv P^{1/2}$, which is also $g$-self-adjoint and $g$-positive definite.

We define the map $r$ by $g \mapsto J_g \equiv Q^{-1} A$. Then $J_g^2 = Q^{-1}AQ^{-1}A = Q^{-2}A^2 = -I$ is a complex structure. To check that $J_g$ is $\omega$-compatible, note that

$$\omega(Q^{-1} Av, Q^{-1} Aw) = g(AQ^{-1} Av, AQ^{-1} Aw) = g(v, Aw) = \omega(v, w), \quad \forall v, w \in V,$$

$$\omega(v, Q^{-1} Av) = g(Av, Q^{-1} Av) > 0 \quad \forall 0 \neq v \in V$$

because $Q^{-1}$ is $g$-self-adjoint and $g$-positive definite.

Finally, for any $\psi \in \text{Sp}(V, \omega)$, replacing $g$ with $\psi^* g$ changes $A$ to $\psi^{-1}A\psi$, and therefore changes $Q$ to $\psi^{-1}Q\psi$. This implies $r(\psi^* g) = \psi^* r(g)$. If $g = g_J$, then $A = J$ and $Q = I$, so that $r(g_J) = J$.

\[ \square \]

2. SYMPLECTIC VECTOR BUNDLES

Definition 2.1. A symplectic vector bundle over a smooth manifold $M$ is a pair $(E, \omega)$, where $E \to M$ is a real vector bundle and $\omega$ is a smooth section of $E^* \wedge E^*$ such that for each $p \in M$, $(E_p, \omega_p)$ is a symplectic vector space. (Here $E^*$ is the dual of $E$.) The section $\omega$ is called a symplectic bilinear form on $E$. Two symplectic vector bundles $(E_1, \omega_1)$, $(E_2, \omega_2)$ are said to be isomorphic if there exists an isomorphism $\phi : E_1 \to E_2$ (which is identity over $M$) such that $\phi^* \omega_2 = \omega_1$.

The standard constructions in bundle theory carry over to the case of symplectic vector bundles. For example, for any smooth map $f : N \to M$ and symplectic vector bundle $(E, \omega)$ over $M$, the pull-back $(f^* E, f^* \omega)$ is a symplectic vector bundle over $N$. In particular, for any submanifold $Q \subset M$, the restriction $(E|_Q, \omega|_Q)$ is a symplectic vector bundle over $Q$. Let $F$ be a sub-bundle of $E$ such that for each
$p \in M$, $(F_p, \omega_p|_{F_p})$ is a symplectic vector space. Then $(F, \omega|_F)$ is naturally a symplectic vector bundle. We call $F$ (or $(F, \omega|_F)$) a **symplectic sub-bundle** of $(E, \omega)$. The **symplectic complement** of $F$ is the sub-bundle

$$F^\omega \equiv \cup_{p \in M} F_p^\omega = \cup_{p \in M} \{ v \in E_p| \omega_p(v, w) = 0, \forall w \in F_p \},$$

which is naturally a symplectic sub-bundle of $(E, \omega)$. Note that as a real vector bundle, $F^\omega$ is isomorphic to the quotient bundle $E/F$.

Given any symplectic vector bundles $(E_1, \omega_1)$, $(E_2, \omega_2)$, the symplectic direct sum $(E_1 \oplus E_2, \omega_1 \oplus \omega_2)$ is naturally a symplectic vector bundle. With this understood, note that for any symplectic sub-bundle $F$ of $(E, \omega)$, one has

$$(E, \omega) = (F, \omega|_F) \oplus (F^\omega, \omega|_{F^\omega}).$$

**Example 2.2.** Let $(M, \omega)$ be a symplectic manifold. Note that $\omega$ as a 2-form on $M$ is a smooth section of $\Omega^2(M) \equiv T^*M \wedge T^*M$. The nondegeneracy condition on $\omega$ implies that $(TM, \omega)$ is a symplectic vector bundle. Note that the closedness of $\omega$ is irrelevant here.

Suppose $Q$ is a symplectic submanifold of $(M, \omega)$. Then $TQ$ is a symplectic sub-bundle of $(TM|_Q, \omega|_Q)$. The normal bundle $\nu_Q \equiv TM|_Q/TQ$ of $Q$ in $M$ is also naturally a symplectic sub-bundle of $(TM|_Q, \omega|_Q)$ by identifying $\nu_Q$ with the symplectic complement $TQ^\omega$ of $TQ$. Notice the symplectic direct sum

$$TM|_Q = TQ \oplus \nu_Q.$$

**Definition 2.3.** Let $(E, \omega)$ be a symplectic vector bundle over $M$. A **complex structure** $J$ of $E$, i.e., a smooth section $J$ of $\text{Aut}(E) \to M$ such that $J^2 = -I$, is said to be $\omega$-**compatible** if for each $p \in M$, $J_p$ is $\omega_p$-compatible, i.e., $J_p \in \mathcal{J}(E_p, \omega_p)$. The space of all $\omega$-compatible complex structures of $E$ is denoted by $\mathcal{J}(E, \omega)$.

**Example 2.4.** Let $(M, \omega)$ be a symplectic manifold. Then a complex structure $J$ on $M$ is simply what we call an almost complex structure on $M$. An almost complex structure $J$ on $M$ is said to be $\omega$-**compatible** if $J \in \mathcal{J}(TM, \omega)$. In this context, we denote $\mathcal{J}(TM, \omega)$, the set of $\omega$-compatible almost complex structures on $M$, by $\mathcal{J}(M, \omega)$. Notice that the closedness of $\omega$ is irrelevant here.

In what follows, we will address the issue of classification of symplectic vector bundles up to isomorphisms, and determine the topology of the space $\mathcal{J}(E, \omega)$.

**Lemma 2.5.** Let $(E, \omega)$ be a symplectic vector bundle over $M$ of rank $2n$.

1. There exists an open cover $\{U_i\}$ of $M$ such that for each $i$, there is a symplectic trivialization $\phi_i : (E|_{U_i}, \omega|_{U_i}) \to (U_i \times \mathbb{R}^{2n}, \omega_0)$. In particular, the transition functions $\phi_{ji}(p) \equiv \phi_j \circ \phi_i^{-1}(p) \in \text{Sp}(2n)$ for each $p \in U_i \cap U_j$, and $E$ becomes a $\text{Sp}(2n)$-vector bundle. Conversely, any $\text{Sp}(2n)$-vector bundle is a symplectic vector bundle, and their classification up to isomorphisms is identical.

2. $(E, \omega)$ as a $\text{Sp}(2n)$-vector bundle admits a lifting to a $U(n)$-vector bundle if and only if there exists a $J \in \mathcal{J}(E, \omega)$.

**Proof.** For any $p \in M$, one can prove by induction as in Theorem 1.4 (with a parametric version) that there exists a small neighborhood $U_p$ of $p$ and smooth sections
$u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ of $E$ over $U_p$ such that for each $q \in U_p$,

$$u_1(q), u_2(q), \cdots, u_n(q), v_1(q), v_2(q), \cdots, v_n(q)$$

form a symplectic basis of $(E_q, \omega_q)$. Part (1) follows immediately from this by defining

$$\phi_p : (E|_{U_p}, \omega|_{U_p}) \to (U_p \times \mathbb{R}^{2n}, \omega_0)$$

as the inverse of

$$(q, (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)^T) \mapsto \sum_{i=1}^{n} (x_i u_i(q) + y_i v_i(q)).$$

For part (2), if $(E, \omega)$ as a Sp$(2n)$-vector bundle admits a lifting to a $U(n)$-vector bundle, then the corresponding complex structure $J$ on $E$ is $\omega$-compatible because the Hermitian structure and $J$ determines $\omega$ completely. If there exists a $J \in \mathcal{J}(E, \omega)$, then one can show that there are local smooth sections $u_1, u_2, \cdots, u_n$ which form a unitary basis at each point for $(E, J, h, f)$. This makes $E$ into a $U(n)$-vector bundle, which is a lifting of the Sp$(2n)$-vector bundle because $u_1, u_2, \cdots, u_n, J u_1, J u_2, \cdots, J u_n$ are local smooth sections which form a symplectic basis at each point for $(E, \omega)$.

**Lemma 2.6.** Any Sp$(2n)$-vector bundle over a smooth manifold admits a lifting to a $U(n)$-vector bundle, which is unique up to isomorphisms (as $U(n)$-vector bundles). Consequently, for any $J_1, J_2 \in \mathcal{J}(E, \omega)$, the complex vector bundles $(E, J_1)$, $(E, J_2)$ are isomorphic. (In other words, every symplectic vector bundle has a underlying complex vector bundle structure unique up to isomorphisms.)

**Proof.** By Theorem 1.12, Sp$(2n)/U(n)$ is contractible. This implies that the classifying spaces $B$Sp$(2n)$ and $BU(n)$ are homotopy equivalent via $i_* : BU(n) \to B$Sp$(2n)$ induced by $i : U(n) \subset$ Sp$(2n)$. In particular, any Sp$(2n)$-vector bundle over a smooth manifold $M$, which is classified by a map from $M$ into $B$Sp$(2n)$ unique up to homotopy, can be lifted to a $U(n)$-vector bundle by lifting the classifying map to a map from $M$ into $BU(n)$, and such a lifting is unique up to isomorphisms.

**Theorem 2.7.** Let $(E_1, \omega_1)$, $(E_2, \omega_2)$ be two symplectic vector bundles. Then they are isomorphic as symplectic vector bundles iff they are isomorphic as complex vector bundles.

**Proof.** Pick $J_1 \in \mathcal{J}(E_1, \omega_1)$, $J_2 \in \mathcal{J}(E_2, \omega_2)$. Then by the previous lemma $(E_1, \omega_1)$, $(E_2, \omega_2)$ are isomorphic as symplectic vector bundles iff $(E_1, J_1, \omega_1)$, $(E_2, J_2, \omega_2)$ are isomorphic as $U(n)$-vector bundles. But the classification of $U(n)$-vector bundles up to isomorphisms is the same as classification of the underlying complex vector bundles because $GL(n, \mathbb{C})/U(n)$ is contractible. The theorem follows immediately.

**Theorem 2.8.** For any symplectic vector bundle $(E, \omega)$, the space of $\omega$-compatible complex structures $\mathcal{J}(E, \omega)$ is nonempty and contractible.

**Proof.** There are actually two proofs of this important fact.

**Proof 1:** The nonemptiness of $\mathcal{J}(E, \omega)$ follows from Lemmas 2.5 and 2.6. On the other hand, for any $J \in \mathcal{J}(E, \omega)$, a parametric version of Corollary 1.14 gives rise to a deformation retraction of $\mathcal{J}(E, \omega)$ to $\{J\}$, which shows that $\mathcal{J}(E, \omega)$ is contractible.
Proof 2: A parametric version of Theorem 1.15 gives rise to a similar map \( r : \text{Met}(E) \to \mathcal{J}(E, \omega) \). Contractibility of \( \mathcal{J}(E, \omega) \) follows from convexity of \( \text{Met}(E) \).

Corollary 2.9. For any symplectic manifold \((M, \omega)\), the space of \(\omega\)-compatible almost complex structures on \(M\) is nonempty and contractible.

Proof 2 of Theorem 2.8 is less conceptual than proof 1 but more useful in various concrete constructions. As an example of illustration, we prove the following Proposition 2.10.

Let \( Q \) be a symplectic submanifold of \((M, \omega)\). Then for any \( J \in \mathcal{J}(Q, \omega|_Q) \), there exists a \( \tilde{J} \in \mathcal{J}(M, \omega) \) such that \( \tilde{J}|_TQ = J \). In particular, every symplectic submanifold of \((M, \omega)\) is a pseudo-holomorphic submanifold for some \(\omega\)-compatible almost complex structure on \(M\).

Proof. Recall the symplectic direct sum decomposition
\[
(TM|_Q, \omega|_Q) = (TQ, \omega|_TQ) \oplus (\nu_Q, \omega|_{\nu_Q}),
\]
where \( \nu_Q \) is the normal bundle of \( Q \) in \( M \). For any \( J \in \mathcal{J}(Q, \omega|_Q) \), we can extend it to \( J' = (J, J') \) by choosing a \( J' \in \mathcal{J}(\nu_Q, \omega|_{\nu_Q}) \). We then extend the corresponding metric \( \omega(\cdot, J' \cdot) \) on \( TM|_Q \) over the whole \( M \) to a metric \( g \) on \( TM \). Let \( r : \text{Met}(M) \to \mathcal{J}(M, \omega) \) be the parametric version of the map in Theorem 1.15. Then \( \tilde{J} \equiv r(g) \) satisfies \( \tilde{J}|_TQ = J \), and in particular, \( Q \) is a pseudo-holomorphic submanifold with respect to the \(\omega\)-compatible almost complex structure \( \tilde{J} \) on \( M \).

We end this section with a brief discussion about integrability of almost complex structures. Recall that an almost complex structure \( J \) on a manifold \( M \) is said to be integrable, if \( M \) is the underlying real manifold of a complex manifold and \( J \) comes from the complex structure.

Let \((M, J)\) be an almost complex manifold with almost complex structure \( J \). The **Nijenhuis tensor** of \( J \) is defined by
\[
\]
for two vector fields \( X, Y : M \to TM \). \( N_J \) is a bilinear map \( T_pM \times T_pM \to T_pM \) for each \( p \in M \), and has properties \( N_J(X, X) = 0 \) and \( N_J(X, JX) = 0 \) for any vector field \( X \). In particular, \( N_J = 0 \) if \( M \) is 2-dimensional. Moreover, one can also check easily that \( N_J = 0 \) if \( J \) is integrable. The converse is given by the following highly nontrivial theorem of A. Newlander and L. Nirenberg.

**Theorem 2.11.** (Newlander-Nirenberg). An almost complex structure is integrable if and only if the Nijenhuis tensor vanishes.

In particular, every symplectic 2-dimensional manifold is Kähler.

**References**