LECTURE 1: BASIC CONCEPTS, PROBLEMS, AND EXAMPLES

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In this lecture we give a general introduction to the basic concepts and some of the fundamental problems in symplectic geometry/topology, where along the way various examples are also given for the purpose of illustration. We will often give statements without proofs, which means that their proof is either beyond the scope of this course or will be treated more systematically in later lectures.

1. Symplectic Manifolds

Throughout we will assume that $M$ is a $C^\infty$-smooth manifold without boundary (unless specific mention is made to the contrary). Very often, $M$ will also be closed (i.e., compact).

**Definition 1.1.** A symplectic structure on a smooth manifold $M$ is a 2-form $\omega \in \Omega^2(M)$, which is (1) nondegenerate, and (2) closed (i.e. $d\omega = 0$). (Recall that a 2-form $\omega \in \Omega^2(M)$ is said to be nondegenerate if for every point $p \in M$, $\omega(u, v) = 0$ for all $u \in T_pM$ implies $v \in T_pM$ equals 0.) The pair $(M, \omega)$ is called a symplectic manifold.

Before we discuss examples of symplectic manifolds, we shall first derive some immediate consequences of a symplectic structure.

1. The nondegeneracy condition on $\omega$ is equivalent to the condition that $M$ has an even dimension $2n$ and the top wedge product

   $\omega^n \equiv \omega \wedge \omega \cdots \wedge \omega$

   is nowhere vanishing on $M$, i.e., $\omega^n$ is a volume form. In particular, $M$ must be orientable, and is canonically oriented by $\omega^n$. The nondegeneracy condition is also equivalent to the condition that $M$ is almost complex, i.e., there exists an endomorphism $J$ of $TM$ such that $J^2 = -\text{Id}$. The latter is homotopy theoretic in nature as it means that the tangent bundle $TM$ as a $SO(2n)$-bundle can be lifted to a $U(n)$-bundle under the natural homomorphism $U(n) \rightarrow SO(2n)$. (More systematic discussion in Lecture 2.) Note that this has nothing to do with the closedness of $\omega$.

2. The closedness of $\omega$ is a very important, nontrivial geometric or analytical condition. For now, let us simply observe that $\omega$ defines a deRham cohomology class $[\omega] \in H^2(M)$, which must be nonzero when $M$ is closed. In fact in this case,

   $[\omega]^n \equiv [\omega] \cup [\omega] \cdots \cup [\omega] \in H^{2n}(M)$

   must be nonzero, since $[\omega]^n([M]) = \int_M \omega^n \neq 0$. As an example, this shows that for $n \geq 2$, the spheres $S^{2n}$ admit no symplectic structures even though they are
all even dimensional and orientable (because $S^{2n}$ is closed and $H^2(S^{2n}) = 0$ when $n \geq 2$). Of course, the failure may simply be caused by nonexistence of nondegenerate 2-forms (or equivalently, nonexistence of almost complex structures) which is homotopy theoretic in nature, rather than by the failure of closedness of the 2-forms which is geometric or analytical in nature. This is almost true, except for the case of $S^6$.

**Example 1.2.** (Calabi, 1958). $S^6$ admits an almost complex structure.

An important fact here is that there is a vector product $\times$ in $\mathbb{R}^7$ (related to Cayley numbers), which is bilinear and skew symmetric, and is related to the standard inner product $(\cdot, \cdot)$ by the following rules:

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle$$

and

$$u \times v \times w + u \times (v \times w) = 2\langle u, w \rangle v - \langle v, w \rangle u - \langle v, u \rangle w.$$ 

The key properties we shall need here are (1) $u \times v$ is orthogonal to both $u, v$, and (2) $u \times (u \times v) = -v$ if $u$ is a unit vector, which can be easily derived from the above rules.

Now for any $x \in S^6 \subset \mathbb{R}^7$, let $\nu(x)$ be the unit outward normal vector of $S^6$ at $x$. We define

$$J_x u = \nu(x) \times u, \quad \forall u \in T_x S^6.$$ 

Then $J_x u \in T_x S^6$ because it is orthogonal to $\nu(x)$, and $J_x^2 = -Id$ because $J_x^2 u = \nu(x) \times (\nu(x) \times u) = -u$ as $\nu(x)$ is a unit vector.

*Note: It is an open question as whether $S^6$ is a complex manifold; the almost complex structure defined above is NOT integrable.*

Here is a fundamental question in symplectic topology.

**Problem 1.3.** (Existence of symplectic structures). Suppose $M$ is an almost complex manifold of dimension $2n$ where $n \geq 2$, and moreover, when $M$ is closed, there exists a cohomology class $a \in H^2(M)$ such that $0 \neq a^n \in H^{2n}(M)$. Does $M$ admit a symplectic structure? If not, what additional assumptions do we need?

Note that for the case of $n = 1$, i.e., $M$ is a Riemann surface, the problem is trivial (and the answer is yes). The case where $M$ is open (without boundary) was solved affirmatively by M. Gromov, which shows that in this case the existence of a symplectic structure is a homopopy theoretic (the so-called “soft”) problem.

**Theorem 1.4.** (Gromov, 1969). Suppose $M$ is an open manifold. Then every nondegenerate 2-form on $M$ (as long as it exists) is homotopic through nondegenerate 2-forms on $M$ to a symplectic structure.

The case where $M$ is closed turns out to be much harder. Gromov’s technique (the so-called “h-principle”) does not work in this case, but for a long time no one has been able to provide a negative example until C. Taubes’ work on the Seiberg-Witten invariants of symplectic 4-manifolds. This work reveals an important connection between the symplectic topology and differential topology of 4-manifolds.
**Theorem 1.5.** (Taubes, 1994). The 4-manifold $M = \mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ is almost complex and has a cohomology class $a \in H^2(M)$ such that $a^2 \neq 0$, however, $M$ does not admit any symplectic structure.

Now we give some natural examples of symplectic manifolds.

**Example 1.6.** (Euclidean Spaces). The most basic examples are the Euclidean spaces $\mathbb{R}^{2n}$ equipped with the standard symplectic structure

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$ 

Show that with $\mathbb{R}^{2n} \cong \mathbb{C}^n$ under $z_j = x_j + iy_j$, $j = 1, 2, \cdots, n$,

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

**Example 1.7.** (Cotangent Bundles). Let $L$ be a smooth manifold of dimension $n$ and $M \equiv T^*L$ be the cotangent bundle. There is a canonical symplectic structure on $M$ which has the form $\omega = -d\lambda$.

The 1-form $\lambda$ is defined as follows. Let $\pi : M \equiv T^*L \rightarrow L$ be the natural projection. Then for any $v \in M \equiv T^*L$, we have $\pi^*: T^*_{\pi(v)}L \rightarrow T_v^*M$. With this understood, $\lambda$ is defined by setting its value at $v$ to be $\lambda(v) \equiv \pi^*(v)$. (Note that $v \in T^*_{\pi(v)}L$, so that $\lambda(v) \equiv \pi^*(v) \in T^*_vM$.)

Let $q_1, \cdots, q_n$ be local coordinates on $L$, and $p_1, \cdots, p_n$ be the corresponding coordinates on the fibers, i.e, if a cotangent vector $v = \sum_{j=1}^{n} p_j dq_j$, then $v$ has coordinates $p_1, \cdots, p_n$ on the fiber. Together $q_1, \cdots, q_n$ and $p_1, \cdots, p_n$ form a system of local coordinates on $M \equiv T^*L$. In the above local coordinates, we claim $\lambda = \sum_{j=1}^{n} p_j dq_j$.

Accepting it momentarily, we see immediately that $\omega = -d\lambda$ is a symplectic structure, because $\omega = \sum_{j=1}^{n} dq_j \wedge dp_j$ in these coordinates.

To see the claim $\lambda = \sum_{j=1}^{n} p_j dq_j$, recall that the value of $\lambda$ at $v = \sum_{j=1}^{n} p_j dq_j$ equals $\pi^*(v) = \sum_{j=1}^{n} p_j \pi^*(dq_j) = \sum_{j=1}^{n} p_j d(\pi \circ q_j)$. Since $q_1, \cdots, q_n$ are regarded as local coordinates on $M \equiv T^*L$, $q_j = \pi \circ q_j$, and with this understood, $v = \sum_{j=1}^{n} p_j dq_j$ has coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$. This shows that $\lambda = \sum_{j=1}^{n} p_j dq_j$. It is interesting to check that $\lambda$ is characterized by the following property: for any 1-form $\sigma$ on $L$, which can be regarded as a section of the cotangent bundle $\pi : M \equiv T^*L \rightarrow L$, one has $\sigma^\ast \lambda = \sigma$.

We remark that cotangent bundles form a fundamental class of symplectic manifolds. These are the phase spaces in classical mechanics, with coordinates $q$ and $p$ corresponding to position and momentum.

**Example 1.8.** (Orientable Surfaces). Let $M$ be a 2-dimensional orientable manifold, and let $\omega$ be any volume form on $M$. Then $(M, \omega)$ is a symplectic manifold because in this case $\omega$ is automatically nondegenerate and closed.

**Example 1.9.** (Kähler Manifolds). Let $M$ be a complex manifold of dimension $n$, and let $h$ be a Hermitian metric on $M$. Then in a local complex coordinates $z_1, z_2, \cdots, z_n$, $h$ may be written as $h = \sum_{j,k=1}^{n} h_{jk} dz_j \otimes d\bar{z}_k$, where $(h_{jk})$ is a $n \times n$ Hermitian matrix, i.e., $h_{jk} = \bar{h}_{kj}$. The associated 2-form to $h$ is $\omega = i \sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\bar{z}_k$. With
this understood, $h$ is called a Kähler metric if $\omega$ is closed. $\omega$ is clearly nondegenerate, hence the underlying real manifold, also denoted by $M$, is a $2n$-dimensional symplectic manifold with a symplectic structure $\omega$. (Note that the nondegeneracy condition on $\omega$ follows from the identity
\[ \omega^n = i^n \det(h_{j\bar{k}}) \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \]
Verify the identity.)

An important aspect of symplectic geometry is its connection with almost Kähler geometry. One can regard $M$ as a $2n$-dimensional real manifold equipped with an (integrable) almost complex structure $J$. In this context $h$ can be regarded as a $J$-invariant complex symmetric bilinear form on the complexification of the almost complex vector bundle $TM$, and $\omega$ and $h$ are related by $h(\cdot, \cdot) = \omega(\cdot, J(\cdot))$.

The complex projective spaces $\mathbb{CP}^n$ form a fundamental class of Kähler manifolds. There is a canonical Kähler metric on $\mathbb{CP}^n$, called the Fubini-Study metric. In terms of the homogeneous coordinates $z_0, z_1, \ldots, z_n$ of $\mathbb{CP}^n$, the associated Kähler form of the Fubini-Study metric is
\[ \omega_0 = \frac{i}{2\pi} \left( \sum_{n=0}^{\infty} z_n \bar{z}_n \right)^2 \sum_{k=0}^{n} \left( z_j z_k dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k \right). \]
(We point out that $\omega_0$ is normalized such that $\int_{\mathbb{CP}^n} \omega_0^n = 1$.) Note that a complex submanifold of $\mathbb{CP}^n$ is naturally a Kähler manifold with the induced metric.

**Example 1.10.** (Product of Symplectic Manifolds). Given two symplectic manifolds $(M_1, \omega_1), (M_2, \omega_2)$, the product $M_1 \times M_2$ is also symplectic with symplectic structures $\pi_1^* \omega_1 \oplus \pi_2^* (\pm \omega_2)$. Here $\pi_i : M_1 \times M_2 \to M_i, i = 1, 2$, and we canonically identify $T^*(M_1 \times M_2)$ with $\pi_1^*(T^*M_1) \oplus \pi_2^*(T^*M_2)$.

A symplectomorphism of a symplectic manifold $(M, \omega)$ is a diffeomorphism $\psi \in \text{Diff}(M)$ which preserves the symplectic structure
\[ \omega = \psi^* \omega. \]
Note that a symplectomorphism is particularly a volume-preserving diffeomorphism (hence is necessarily an orientation-preserving diffeomorphism), as one has $\psi^*(\omega^n) = (\psi^* \omega)^n = \omega^n$. There is an abundance of symplectomorphisms on a symplectic manifold. In fact, the group of symplectomorphisms, denoted by $\text{Symp}(M, \omega)$ or simply by $\text{Symp}(M)$, is an infinite dimensional Lie group, in contrast with the finite dimensionality of the isometry group of a Riemannian metric.

It has been an open question as whether symplectomorphisms differ from volume-preserving diffeomorphisms, until in 1985 M. Gromov published the revolutionary paper on pseudoholomorphic curves in symplectic manifolds, where he proved the following celebrated non-squeezing result.

**Theorem 1.11.** (Gromov, 1985). One can not squeeze the unit ball $\mathbb{B}^{2n}(1) \subset \mathbb{R}^{2n}$ into a “long and thin” cylinder $\mathbb{B}^2(r) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ (where $r < 1$) via an embedding which preserves the symplectic structure $\omega$ on $\mathbb{R}^{2n}$, however, one can always squeeze the ball into the “long and thin” cylinder if only the volume form $\omega_0^n$ is required to be preserved.
The nondegeneracy condition of a symplectic structure $\omega$ gives rise to the following canonical isomorphism

$$TM \to T^*M : X \mapsto \iota(X) \omega = \omega(X, \cdot).$$

A vector field $X$ is called symplectic if $\iota(X) \omega$ is closed. The next result is one consequence of the closedness of a symplectic structure, which shows that when $M$ is closed, the set of symplectic vector fields form the Lie algebra of the group $\text{Symp}(M)$.

**Proposition 1.12.** Let $M$ be a closed manifold. If $t \mapsto \psi_t \in \text{Diff}(M)$ is a smooth family of diffeomorphisms generated by a family of vector fields $X_t$ via

$$\frac{d}{dt} \psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id},$$

then $\psi_t \in \text{Symp}(M)$ for every $t$ iff $X_t$ is a symplectic vector field for every $t$.

**Proof.** Recall Cartan’s formula for the Lie derivative

$$\mathcal{L}_X \omega = \iota(X) d\omega + d(\iota(X) \omega).$$

Now the closedness of $\omega$, i.e., $d\omega = 0$, implies that

$$\frac{d}{dt} \psi_t^* \omega = \psi_t^* (\mathcal{L}_X \omega) = \psi_t^* (d(\iota(X_t) \omega),$$

which vanishes if and only if $\iota(X_t) \omega$ is closed. This proves that $\psi_t \in \text{Symp}(M)$ for every $t$ iff $X_t$ is a symplectic vector field for every $t$. \qed

There is a simple way to obtain symplectic vector fields, and therefore to obtain symplectomorphisms, which shows their abundance. Let $H$ be a smooth function (which has a compact support if $M$ is not closed). There is a vector field $X_H$ canonically associated to $H$ by the following equation

$$\iota(X_H) \omega = dH.$$

The flow $\psi_H^t$ on $M$ generated by $X_H$, i.e.,

$$\frac{d}{dt} \psi_H^t = X_H \circ \psi_H^t,$$

is called a Hamiltonian flow. Note that $X_H$ is a symplectic vector field as $\iota(X_H)$, being exact, is closed. Thus $\psi_H^t$ is a symplectomorphism for each $t$. The function $H$ is called the Hamiltonian function and the vector field $X_H$ is called the Hamiltonian vector field.

More generally, a symplectomorphism $\psi \in \text{Symp}(M)$ is called Hamiltonian if there exists a smooth family of $\psi_t \in \text{Symp}(M)$, $t \in [0, 1]$, with $\psi_0 = \text{id}$, $\psi_1 = \psi$, such that the corresponding (time-dependent) symplectic vector field $X_t$ generating $\psi_t$ is Hamiltonian, i.e., the 1-form $\iota(X_t) \omega$ is exact for each $t$ and has the form $\iota(X_t) \omega = dH_t$ for a time-dependent smooth function $H_t$ on $M$. The function $H_t$ is called a time-dependent Hamiltonian function, and $\psi_t$ is called an Hamiltonian isotopy.
Example 1.13. (A Hamiltonian $S^1$-Action). Consider the symplectic manifold $(M, \omega)$ where $M$ is the unit sphere in $\mathbb{R}^3$, i.e.

$$M = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = 1\},$$

and $\omega$ is the area form induced from the embedding $M \subset \mathbb{R}^3$. We will show that the Hamiltonian flow on $M$ associated to the “height” function

$$H(x_1, x_2, x_3) \equiv x_3$$

is the $S^1$-action on $M$ defined by the rotation about the $x_3$-axis.

To see this, we change $x_1, x_2$ into polar coordinates $r, \theta$, and note that in terms of $r, \theta, x_3$ the volume form of $\mathbb{R}^3$ is $rdr \wedge d\theta \wedge dx_3$. On the unit sphere $M$, $r = 1$ so that the restriction of the volume form on $M$ is $v = dr \wedge d\theta \wedge dx_3$. This implies that

$$\omega = \iota(\frac{\partial}{\partial r})v = d\theta \wedge dx_3.$$}

Recall that the Hamiltonian vector field $X_H$ is defined by $\iota(X_H)\omega = dH$. It follows that for $H = x_3$, $X_H = \frac{\partial}{\partial \theta}$, and therefore the associated Hamiltonian flow is given by the rotation about the $x_3$-axis.

Question 1.14. If a symplectic $S^1$-action has a fixed point, is it necessarily Hamiltonian? Is there a characterization of Hamiltonian $S^1$-actions in terms of fixed points data?

Now we go back to the discussion of examples of symplectic manifolds. Note that the examples we gave earlier of closed symplectic manifolds are all coming from Kähler manifolds. (The example of orientable surfaces is Kähler because every almost complex structure in dimension 2 is integrable.) The following problem has been a basic scheme in symplectic topology, much of which has become known only in the last ten years.

Problem 1.15. (Symplectic $>$ Kähler). How much larger is the category of symplectic manifolds than the category of Kähler manifolds?

Consider the following way of constructing symplectic manifolds. Suppose $(M, \omega)$ is a symplectic manifold and a discrete group $G$ acts on $M$ via symplectomorphisms, i.e., for every $g \in G$, the diffeomorphism $g : M \to M$ is an element of $\text{Symp}(M, \omega)$. Suppose furthermore that the action of $G$ is free, and hence the quotient $M/G$ is again a smooth manifold. Then $\omega$ descends to a symplectic structure on the quotient manifold $M/G$, with which $M/G$ naturally becomes a symplectic manifold. For a typical example, consider $\mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$, and the free action by translation of $G = \mathbb{Z}^{2n}$ the integral lattice. The quotient is the $(2n)$-dimensional torus $\mathbb{T}^{2n}$, with a standard symplectic structure.

The following example is also of this sort, which gives the first example of a closed, symplectic but non-Kähler manifold. (The example was known to Kodaira in the 1950s and was rediscovered in the 1970s by Thurston.)
**Example 1.16.** (A Non-Kähler Manifold). Consider the group $G = \mathbb{Z}^2 \times \mathbb{Z}^2$ with the noncommutative group operation

$$(j', k') \circ (j, k) = (j + j', A_j k + k'), \quad A_j = \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix},$$

where $j = (j_1, j_2) \in \mathbb{Z}^2$, and similarly for $k$. This group acts on $\mathbb{R}^4$ via

$$G \to \text{Diff}(\mathbb{R}^4) : (j, k) \mapsto \rho_{jk},$$

where $\rho_{jk}(x, y) = (x + j, A_j y + k)$. One can verify easily that the action is free and preserves the symplectic structure

$$\omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2.$$

Hence the quotient $M = \mathbb{R}^4 / G$ is a symplectic manifold which is easily seen to be closed.

We will show that $H_1(M; \mathbb{Z}) = \mathbb{Z}^3$ so that $M$ is not a Kähler manifold. (Recall that by the Hodge theory the odd dimensional Betti numbers of a closed Kähler manifold must be even.) To see this, one first verifies that the commutator subgroup $[G, G]$, which consists of elements of the form $aba^{-1}b^{-1}$, $a, b \in G$, equals $0 \oplus 0 \oplus \mathbb{Z} \oplus 0$. Then note that $\pi_1(M) = G$ and $H_1(M; \mathbb{Z}) = \pi_1(M) / [\pi_1(M), \pi_1(M)]$. Hence $H_1(M; \mathbb{Z}) = G / [G, G] = \mathbb{Z}^3$. This implies that the first Betti number of $M$ equals 3, and that $M$ is not Kähler.

Topologically, $M$ is a nontrivial $T^2$ bundle over $T^2$, or more precisely, $M = S^1 \times N$ where $N$ is the nontrivial $T^2$ bundle over $S^1$, defined as follows. Let $x, y$ be the standard coordinates on $T^2$. Then $N = [0, 1] \times T^2 / \sim$, where $(0, x, y) \sim (1, x + y, y)$.

2. Submanifolds of Symplectic Manifolds

Let $(M, \omega)$ be a symplectic manifold, and let $Q \subset M$ be a submanifold of $M$. Note that the tangent bundle $TQ$ is a sub-bundle of $TM|_Q$. We set

$$TQ^\omega \equiv \bigcup_{q \in Q} \{ u \in TM_q | \omega(u, v) = 0 \ \forall v \in TQ_q \},$$

which is also a sub-bundle of $TM|_Q$.

We say $Q$ is **isotropic** if $TQ \subset TQ^\omega$, **coisotropic** if $TQ^\omega \subset TQ$, **symplectic** if $TQ \cap TQ^\omega = \{0\}$, and **Lagrangian** if $TQ = TQ^\omega$. Amongst these four types of submanifolds of a symplectic manifold, Lagrangian submanifolds form the most important class, which is what we shall concentrate in this section. Note that a Lagrangian submanifold in a $2n$-dimensional symplectic manifold has dimension $n$.

**Example 2.1.** (Totally real submanifolds in a Kähler manifold). Let $M$ be a Kähler manifold of complex dimension $n$. A real $n$-dimensional submanifold $Q$ is called **totally real** if for every point $q \in Q$ there is a local holomorphic complex coordinates centered at $q$ within which $Q$ is identified with the real part $\mathbb{R}^n \subset \mathbb{C}^n$. Check that a totally real submanifold in a Kähler manifold is Lagrangian with respect to the Kähler form. When $n = 1$, every symplectic manifold is Kähler, and every real 1-dimensional submanifold is Lagrangian and totally real. For higher dimensional examples, suppose $M$ is a complex submanifold of $\mathbb{CP}^N$, which is Kähler under the induced metric. Then the
fixed-point set of the anti-holomorphic involution \( z_i \mapsto \bar{z}_i \) in \( M \), if nonempty, is a totally real submanifold of \( M \).

**Example 2.2.** (Lagrangian submanifolds in \( (\mathbb{R}^{2n}, \omega_0) \)). The \( n \)-dimensional spaces defined by \( x_j = \text{constant}, j = 1, \ldots, n \), or \( y_j = \text{constant}, j = 1, \ldots, n \), are Lagrangian submanifolds in \( (\mathbb{R}^{2n}, \omega_0) \). For a different type of examples, consider the \( n \)-torus \( T^n \subset \mathbb{R}^{2n} \), where we think \( (\mathbb{R}^{2n}, \omega_0) \) as a product of symplectic manifold \( (\mathbb{R}^2, \omega_0) \) and

\[
T^n = S^1 \times \cdots \times S^1,
\]

where the \( j \)-th copy of \( S^1 \) is the unit circle \( \{x_j^2 + y_j^2 = 1\} \) in the \( j \)-th copy of \( \mathbb{R}^2 \). The torus \( T^n \) is Lagrangian because (1) any 1-dimensional manifold in a 2-dimensional symplectic manifold is Lagrangian, and (2) the product of Lagrangian submanifolds in a product symplectic manifold is again Lagrangian.

**Example 2.3.** (Lagrangian submanifolds and symplectomorphisms). Let \((M, \omega)\) be any symplectic manifold. Then in the product symplectic manifold \((M \times M, \omega \times (-\omega))\), the diagonal \( \Delta \equiv \{(x, x) \in M \times M | x \in M\} \) is Lagrangian. More generally, for any \( \psi \in \text{Symp}(M, \omega) \), the graph of \( \psi \) in \( M \times M \),

\[
\text{graph}(\psi) \equiv \{(x, \psi(x)) \in M \times M | x \in M\},
\]

is a Lagrangian submanifold.

**Example 2.4.** (Lagrangian submanifolds in cotangent bundles). Let \( M \equiv T^*L \) be the cotangent bundle of \( L \) equipped with the canonical symplectic structure \( \omega = -d\lambda \). (In local coordinates \( \lambda = \sum_j p_j dq_j \) and \( \omega = \sum_j dq_j \wedge dp_j \).) Clearly, the fibers of \( T^*L \) (defined by \( q_j = \text{constant}, \forall j \)) and the zero section \( L \subset T^*L \) (defined by \( p_j = 0, \forall j \)) are Lagrangian submanifolds.

Next we consider submanifolds \( Q_{\sigma} \) in \( M \) which is the graph of a 1-form \( \sigma \) on \( L \), regarded as a smooth section of \( T^*L \). Since \( Q_{\sigma} \) is of half dimension of \( M \), it follows that \( Q_{\sigma} \) is a Lagrangian submanifold if and only if the pull-back of \( \omega \) to \( Q_{\sigma} \) equals 0, which is equivalent to the condition that the pull-back \( \sigma^*\omega = 0 \). But

\[
\sigma^*\omega = \sigma^*(-d\lambda) = -d(\sigma^*\lambda) = -d\sigma,
\]

which implies that \( Q_{\sigma} \) is Lagrangian iff \( \sigma \) is closed.

One of the fundamental questions in symplectic topology concerns the intersection of Lagrangian submanifolds. In order to motivate this question, we consider the intersection of \( Q_{\sigma} \) for a nonzero closed 1-form \( \sigma \) with the zero section \( Q_0 = L \). Observe that the set of intersection points of \( Q_{\sigma} \) and \( L \) is precisely the set of zeroes of \( \sigma \) on \( L \). With this understood, one should distinguish the following two cases: (1) \( \sigma = df \) is exact, (2) \( \sigma \) is not exact. In the first case, \( Q_{\sigma} \) is called an **exact** Lagrangian submanifold, and the function \( f : L \to \mathbb{R} \) called a **generating function** of \( Q_{\sigma} \). Now the basic observation is that on a closed manifold exact 1-forms \( \sigma = df \) have more zeroes (which correspond to the critical points of the function \( f \)) than closed 1-forms in general. For instance, on the 2-torus \( \mathbb{T}^2 \), any smooth function has at least 3 critical points and any Morse function has at least 4 critical points (which equals the sum of the Betti numbers), while there are closed 1-forms on \( \mathbb{T}^2 \) which are nowhere vanishing.
Lemma 2.5. Suppose \( \sigma \) is a closed 1-form on \( L \) and \( Q_\sigma \) is the corresponding Lagrangian submanifold in \( T^*L \). Let \( \psi_t \in \text{Symp}(T^*L) \), \( t \in [0,1] \), be an isotopy of symplectomorphisms such that \( \psi_0 = \text{id} \) and \( \psi_1(L) = Q_\sigma \). Then \( \sigma \) is exact if \( \psi_t \) is an Hamiltonian isotopy.

Proof. Let \( X_t \) be the corresponding time-dependent symplectic vector field. Then

\[
\frac{d}{dt} \psi_t^* \lambda = \psi_t^* (L_{X_t} \lambda) = \psi_t^* (\iota(X_t)d\lambda + d(\iota(X_t)\lambda)),
\]

which implies that \( \psi_t^* \lambda - \lambda \) is exact for each \( t \) if \( \psi_t \) is an Hamiltonian isotopy, i.e., \( \iota(X_t)\omega = \iota(X_t)d\lambda \) is exact for every \( t \). The lemma follows by observing that, since \( \psi_1(L) = Q_\sigma \),

\[
\sigma = \sigma^* \lambda = i^*(\psi_1^* \lambda - \lambda),
\]

where \( i : L \to T^*L \) is the zero section. \( \square \)

Problem 2.6. (Arnold’s conjecture on Lagrangian intersections). Let \( L_0, L_1 \) be compact Lagrangian submanifolds of a symplectic manifold \( (M, \omega) \). Moreover, suppose that \( \psi_t \in \text{Symp}(M, \omega) \), \( t \in [0,1] \), is an Hamiltonian isotopy such that \( \psi_0 = \text{id} \) and \( L_1 = \psi_1(L_0) \). Then \( L_0 \) and \( L_1 \) must have at least as many intersection points as a function on \( L_0 \) must have critical points.

Let \( (M, \omega) \) be a closed symplectic manifold, and let \( \psi \in \text{Symp}(M, \omega) \). Then as we have seen earlier that both the diagonal \( \Delta \) and the graph of \( \psi \) are Lagrangian submanifolds of the product \( (M \times M, \omega \times (-\omega)) \). Moreover, observe that the intersection points of \( \Delta \) and the graph of \( \psi \) are exactly the fixed points of \( \psi \) in \( M \), i.e., \( x \in M \) such that \( \psi(x) = x \). Now suppose \( \psi_t \) is an Hamiltonian isotopy on \( M \). Then \( \Psi_t \equiv (\text{id}, \psi_t) \) is an Hamiltonian isotopy on \( M \times M \), and \( \Psi_t(\Delta) \) is the graph of \( \psi_t \) for every \( t \).

Problem 2.7. (Arnold’s conjecture on symplectic fixed points). Let \( \psi : M \to M \) be an Hamiltonian symplectomorphism of a closed symplectic manifold \( (M, \omega) \). Then \( \psi \) must have at least as many fixed points as a function on \( M \) must have critical points.

Note that Arnold’s conjecture on symplectic fixed points is trivially true when the Hamiltonian symplectomorphism \( \psi \) is generated by a time-independent Hamiltonian \( H \) on \( M \). Indeed, in this case, the symplectic fixed points are precisely the critical points of the Hamiltonian function \( H \).

We remark that Arnold’s conjectures have been one of the focal points of much of the recent development in symplectic topology. The celebrated Floer Homology theory was initially introduced to tackle these conjectures!

3. Contact Manifolds

Contact topology is the odd-dimensional analogue of symplectic topology. While contact topology is an interesting subject of its own right, the interplay between symplectic topology and contact topology has become a trend in recent years.

Definition 3.1. Let \( N \) be a \((2n+1)\)-dimensional manifold, and let \( \xi \subset TN \) be a hyperplane field, which is defined by a 1-form \( \alpha \), i.e., \( \xi = \ker \alpha \). \( \xi \) is called a contact
structure if \( d\alpha \) is nondegenerate when restricted to \( \xi \), or equivalently, \( \alpha \wedge (d\alpha)^n \neq 0 \). The 1-form \( \alpha \) is called a contact form associated to \( \xi \), and the pair \((N, \xi)\) is called a contact manifold.

**Remark 3.2.** (1) Recall that the Frobenius integrability theorem says that a hyperplane field \( \xi = \ker \alpha \) is integrable if and only if \( \alpha \wedge d\alpha = 0 \). So a contact structure may be thought of as a maximally nonintegrable distribution.

(2) Note that \( \alpha \wedge (d\alpha)^n \) is a volume form on \( N \), in particular, \( N \) is orientable, and is canonically oriented by \( \alpha \wedge (d\alpha)^n \). On the other hand, \( d\alpha \) is nondegenerate on \( \xi \), so that \( \xi \) is canonically oriented by \( (d\alpha)^n \). These together determines an orientation of the quotient bundle \( TN/\xi \). In other words, the contact structure \( \xi \) is both oriented and co-oriented.

(3) Suppose \( \alpha' \) is another 1-form which defines \( \xi \), i.e., \( \xi = \ker \alpha' \). Then \( \alpha' = f\alpha \) where \( f \) is a nowhere vanishing function. One can easily check that
\[
\alpha' \wedge (d\alpha')^n = f^{n+1} \alpha \wedge (d\alpha)^n \neq 0.
\]
Hence \( \alpha' \) is another contact form associated to \( \xi \), and \( \xi \) being a contact structure is independent of the choice of its defining 1-form.

We emphasize that the contact structure is the hyperplane field \( \xi \) not the contact form \( \alpha \). It determines the contact forms up to a nonzero factor \( f \). We usually assume that \( f > 0 \) so that the relevant orientations of the contact structure defined through the contact forms are compatible.

**Definition 3.3.** Let \((N, \xi)\) be a contact manifold of dimension \( 2n + 1 \), and let \( L \) be a \( n \)-dimensional submanifold of \( N \). \( L \) is called Legendrian if \( TL \subset \xi|_L \).

In a certain sense Legendrian submanifolds are contact analogues of Lagrangian submanifolds. Suppose \( \alpha \) is any contact form associated to a contact structure \( \xi \), and suppose \( L \) is a Legendrian submanifold. Then \( d\alpha = 0 \) on \( TL \). To see this, let \( X, Y \) be any vector fields on \( L \). Then \([X, Y]\) is also a vector field on \( L \). Since \( TL \subset \xi|_L \), we have \( \alpha(X) = \alpha(Y) = \alpha([X, Y]) = 0 \), which implies that \( d\alpha(X, Y) = 0 \).

**Example 3.4.** (Euclidean spaces). Let \( x_1, \cdots, x_n, y_1, \cdots, y_n, z \) be the coordinates on \( \mathbb{R}^{2n+1} \). Then the 1-form
\[
\alpha_0 = dz - \sum_{j=1}^{n} y_j dx_j
\]
is a contact form on \( \mathbb{R}^{2n+1} \). Let’s visualize the corresponding contact structure \( \xi_0 \) for the case of \( \mathbb{R}^3 \). Let \( x, y, z \) be the coordinates on \( \mathbb{R}^3 \). Then \( \alpha_0 = dz - y dx \). It is easily seen that
\[
\xi_0 = \bigcup_{(x, y, z) \in \mathbb{R}^3} \{ a\partial_y + b(\partial_x + y\partial_z) | a, b \in \mathbb{R} \}.
\]

**Example 3.5.** (1-jet bundles). Let \( L \) be a \( n \)-dimensional manifold, and let \( N \equiv T^*L \times \mathbb{R} \) be the 1-jet bundle over \( L \). Then
\[
\alpha = dz - \lambda
\]
is a contact form on $N$, where $z$ is the coordinate on the $\mathbb{R}$ factor and $\lambda$ is the canonical 1-form on the $T^*L$ factor. Note that for any smooth function $f : L \to \mathbb{R}$, the submanifold
\[ L_f \equiv \{(x, df(x), f(x)) | x \in L\} \subset N \]
is Legendrian.

**Example 3.6.** (Circle bundles). Let $N$ be a circle bundle over $\Sigma$. Then for any connection 1-form $A \in \Omega^1(N; i\mathbb{R})$ of the bundle, $F_A = dA$ descents to a 2-form on $\Sigma$ such that the first Chern class of the circle bundle $\pi : N \to \Sigma$ is represented by $iF_A/2\pi$. Now suppose that the first Chern form of the circle bundle $\pi : N \to \Sigma$ is represented by a 2-form $\omega$ which is a symplectic structure on $\Sigma$. Then one can choose a connection 1-form $A$ such that $F_A = -2\pi i\pi^*\omega$. We set
\[ \alpha \equiv \frac{1}{2\pi} A. \]
Then $\alpha$ is a contact 1-form on $N$ with $d\alpha = \pi^*\omega$. One basic example of the above construction is given by the Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$, where the first Chern class is represented by the Kähler form of the Fubini-Study metric on $\mathbb{C}P^n$.

Let $(N, \xi)$ be a contact manifold and $\alpha$ be a contact form associated to the contact structure $\xi$. A diffeomorphism $\psi \in \text{Diff}(N)$ is called a contactomorphism if $\psi$ preserves the oriented hyperplane field $\xi$, or equivalently, if there is a smooth function $h$ such that $\psi^*\alpha = e^h\alpha$. A contact isotopy is a smooth family $\psi_t$ of contactomorphisms such that $\psi_t^*\alpha = e^{ht}\alpha$ for a smooth family of functions $h_t$.

A vector field $X$ on a contact manifold $(N, \xi)$ is called a contact vector field if $\mathcal{L}_X\alpha = ga$ for a contact form $\alpha$ associated to $\xi$, where $g$ is a smooth function on $N$. Consider a time-dependent contact vector field $X_t$, such that $\mathcal{L}_{X_t}\alpha = g_t\alpha$, and the smooth family of diffeomorphisms $\psi_t$ generated by $X_t$, i.e.,
\[ \frac{d}{dt} \psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id}. \]
One can check easily that $\psi_t$ is a contact isotopy. In fact, since
\[ \frac{d}{dt} \psi_t^*\alpha = \psi_t^* \mathcal{L}_{X_t}\alpha = \psi_t^* g_t\alpha, \]
one has $\psi_t^*\alpha = e^{ht}\alpha$ where $h_t = \int_0^t \psi_s^* g_s ds$. Conversely, if $\psi_t$ is a contact isotopy with $\psi_t^*\alpha = e^{ht}\alpha$, then $\mathcal{L}_{X_t}\alpha = g_t\alpha$ where $g_t = (\psi_t^{-1})^* \frac{d}{dt} h_t$.

Given any contact form $\alpha$, there is a canonical contact vector field $X_\alpha$, called a Reeb vector field, which has the property $\mathcal{L}_{X_\alpha}\alpha = 0$. $X_\alpha$ is uniquely determined by the conditions
\[ \iota(X_\alpha) d\alpha = 0 \quad \text{and} \quad \iota(X_\alpha) d\alpha = 1. \]
The dynamical system generated by a Reeb vector field is called a Reeb dynamics. More generally, fixing a contact form $\alpha$, there is a 1-1 correspondence between contact vector fields and smooth functions as follows:
\[ \iota(X_H) \alpha = -H \quad \text{and} \quad \iota(X_H) d\alpha = dH - (\iota(X_\alpha) dH) \alpha, \]
where $H$ is a smooth function and $X_H$ is the corresponding contact vector field. Note that $X_\alpha$ corresponds to $H \equiv -1$. 
We close this section with a discussion on the interaction between contact topology and symplectic topology.

Recall that symplectic geometry is the geometry on which Hamiltonian mechanics rests. One of the fundamental problems in Hamiltonian mechanics concerns the existence of periodic solutions, i.e., periodic orbits of the corresponding Hamiltonian dynamical system. More concretely, suppose \((M, \omega)\) is a symplectic manifold and \(H\) is a Hamiltonian function on \(M\) such that each level surface \(H^{-1}(c)\) is compact. Let \(X_H\) be the corresponding Hamiltonian vector field, defined by

\[
\iota(X_H)\omega = dH.
\]

Note that the Hamiltonian dynamical system generated by \(X_H\) preserves the level surfaces of \(H\) as seen easily from the fact \(X_H(H) = \omega(X_H, X_H) = 0\). In particular, the flow \(\psi_t\) defined by

\[
\frac{d}{dt}\psi_t = X_H \circ \psi_t, \quad \psi_0 = \text{id}
\]

exists for \(t \in (-\infty, \infty)\) since each level surface \(H^{-1}(c)\) is compact. A periodic orbit of the Hamiltonian dynamical system is a map \(\gamma: \mathbb{R} \to M\) which satisfies

\[
\frac{d}{dt}\gamma(t) = X_H(\gamma(t)), \quad \forall t \in \mathbb{R}, \quad \text{and} \quad \gamma(T) = \gamma(0) \text{ for some } T > 0.
\]

Note that every periodic orbit is contained in some level surface of \(H\).

Given any contact manifold \((N, \xi)\) with a contact form \(\alpha\), there is a canonical symplectic structure on \(\mathbb{R} \times N\) defined as follows: let \(t\) be the coordinate on the \(\mathbb{R}\) factor, then the 2-form \(d(e^t \alpha)\) is a symplectic structure on \(\mathbb{R} \times N\). The symplectic manifold \((\mathbb{R} \times N, d(e^t \alpha))\) is called the symplectization of \((N, \alpha)\). Note that the vector field \(\partial_t\) on \(\mathbb{R} \times N\) has the property that \(L_{\partial_t} \omega = \omega\) where \(\omega \equiv d(e^t \alpha)\) is the canonical symplectic structure on \(\mathbb{R} \times N\). Such a vector field is called a Liouville vector field.

Let \((M, \omega)\) be a symplectic manifold, and let \(N \subset M\) be a compact hypersurface. We say that \(N\) is of contact type if there exists a contact form \(\alpha\) on \(N\) such that a neighborhood of \(N\) in \((M, \omega)\) is identified symplectically with a neighborhood of \(\{0\} \times N\) in the symplectization of \((N, \alpha)\). For example, the unit sphere \(S^{2n-1}\) in \((\mathbb{R}^{2n}, \omega_0)\) is a hypersurface of contact type with contact form

\[
\alpha_0 = \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j).
\]

**Problem 3.7.** (Weinstein’s conjecture on Hamiltonian dynamics). Suppose a level surface \(H^{-1}(c)\) is of contact type in \((M, \omega)\). Then there exists a periodic orbit of \(X_H\) contained in \(H^{-1}(c)\).

Note that suppose \(H^{-1}(c)\) is of contact type and \(\alpha\) is the contact form on \(H^{-1}(c)\) with respect to which a neighborhood of \(H^{-1}(c)\) in \((M, \omega)\) is identified symplectically with a neighborhood of \(\{0\} \times H^{-1}(c)\) in the symplectization. Then it is easily seen that \(\omega|_{H^{-1}(c)} = d\alpha\). In particular, \(X_H\) is in the null direction of \(d\alpha\), i.e., \(\iota(X_H)d\alpha = 0\). Notice that the periodic orbits of \(X_H\) are in 1-1 correspondence with the periodic orbits of the Reeb vector field \(X_\alpha\) on \(H^{-1}(c)\).
Problem 3.8. (Weinstein’s conjecture on Reeb dynamics). Let $N$ be a compact contact manifold with contact form $\alpha$. Then there exists at least one periodic orbit of the Reeb vector field $X_\alpha$.

For another aspect of contact topology in symplectic geometry, let’s consider symplectic structures on a manifold $M$ with boundary. Suppose $M$ is almost complex, or equivalently, $M$ admits a nondegenerate 2-form. Then by Gromov’s theorem the interior of $M$ admits a symplectic structure. Of course, in general one has no a priori knowledge about the behavior of the symplectic structure near the boundary $\partial M$.

Let $M$ be a manifold with boundary, and let $\omega$ be a symplectic structure on $M$. We say $(M, \omega)$ is a symplectic manifold with contact boundary if $\partial M$ is of contact type and a neighborhood of $\partial M$ in $(M, \omega)$ is symplectically identified with $(-\epsilon, 0] \times \partial M$ in the symplectization for some $\epsilon > 0$. For example, the unit ball $B^{2n}(1) \subset \mathbb{R}^{2n}$ with the standard symplectic structure $\omega_0$ is a symplectic manifold with contact boundary.

While Gromov’s work shows that open manifolds have a rather trivial symplectic topology, closed symplectic manifolds exhibit many rigidity properties, discovered mainly through his powerful theory of pseudoholomorphic curves. Symplectic manifolds with contact boundary resemble in many aspects closed symplectic manifolds, because the pseudoholomorphic curve theory can be properly extended to this context.

Symplectic geometry also provides a useful tool in the study of contact topology, especially in dimension 3. A compact contact manifold $(N, \xi)$ is called symplectically fillable if $(N, \xi)$ can be realized as the boundary of a symplectic manifold with contact boundary. For example, the contact structure on $S^3$ obtained via the Hopf fibration $S^3 \to \mathbb{C}P^1$ as described in Example 3.6 is symplectically fillable; it can be realized as the boundary of the symplectic manifold with contact boundary $(B^4(1), \omega_0)$. Symplectically fillable contact structures on a 3-manifold belong to a class of contact structures, called tight contact structures, which exhibit many rigidity properties.

References