Math 235 Practice Midterm 2 Solutions

Q1.

(a) Perform row reduction to $[A \; I_3]$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ -5 & -1 & 9 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & -1 & -1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 7/3 & 2/3 & 2/3 \\ 0 & 1 & 0 & -17/3 & -1/3 & -4/3 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 \end{bmatrix}.$$

Hence $A^{-1} = \begin{bmatrix} 7/3 & 2/3 & 2/3 \\ -17/3 & -1/3 & -4/3 \\ 2/3 & 1/3 & 1/3 \end{bmatrix}$

(b) $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$, so the solution is

$$\vec{x} = \begin{bmatrix} 7/3 & 2/3 & 2/3 \\ -17/3 & -1/3 & -4/3 \\ 2/3 & 1/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 7 & 2 & 2 \\ -17 & -1 & -4 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 7\times 1 + 2\times 2 + 2\times(-4) \\ -17\times 1 + (-1)\times 2 + (-4)\times(-4) \\ 2\times 1 + 1\times 2 + 1\times(-4) \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

(c) $A^{-1}(B-X)C = I_n \Rightarrow (B-X)C = A \Rightarrow$

$B - X = AC^{-1} \Rightarrow -X = AC^{-1} - B \Rightarrow \boxed{X = B - AC^{-1}}$

Q2.

(a) $\det T = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

$= (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c+a)-(b+a) \end{vmatrix}$

$= (b-a)(c-a) \cdot \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)$.

(b) We show $\det A = 0$, So there are no values of a, b, c Such that A is invertible:

Here we add 2nd row to 3rd row

$\det A = \det A^T = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$

$= (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (a+b+c) \cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix}$

$= (a+b+c) \times 0 = 0$. So $\det A = 0$.

Q3.

(a) Let A be the 3×3 matrix whose columns are the 3 vectors given: $(1,1,1)$, $(2,3,4)$, $(1,1,5)$,

i.e, $A=\begin{bmatrix}1&2&1\\1&3&1\\1&4&5\end{bmatrix}$, then Volume$(P)=|\det A|$.

We compute $\det A$:

$$\begin{vmatrix}1&2&1\\1&3&1\\1&4&5\end{vmatrix}=\begin{vmatrix}1&2&1\\0&1&0\\0&2&4\end{vmatrix}=1\cdot\begin{vmatrix}1&0\\2&4\end{vmatrix}=1\times4-2\times0=4.$$

Hence Volume$(P)=4$.

(b) $$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_1+x_2-2x_3\\3x_2-x_3\\5x_3\end{bmatrix}=\begin{bmatrix}1&1&-2\\0&3&-1\\0&0&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

So $T:\mathbb{R}^3\to\mathbb{R}^3$ is defined by the matrix $B=\begin{bmatrix}1&1&-2\\0&3&-1\\0&0&5\end{bmatrix}$.

By the formula Volume$(T(p))=|\det B|\cdot$Volume(p),

We have Volume$(T(p))=$

$$\left|\det\begin{bmatrix}1&1&-2\\0&3&-1\\0&0&5\end{bmatrix}\right|\cdot\text{Volume}(p)=(1\times3\times5)\cdot 4$$

$$=60.$$

Q4.

(a) $S = \left\{ \begin{bmatrix} a \\ 0 \\ a \\ b \end{bmatrix} : a, b \text{ are in } \mathbb{R} \right\}$.

Since $\begin{bmatrix} a \\ 0 \\ a \\ b \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, we see that

$S = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, hence S is a subspace.

(b) $T(ax^2+bx+c) = a-c$. We find the kernel of T first. Suppose ax^2+bx+c belongs to the Kernel. Then $T(ax^2+bx+c)=0$, which implies $a-c=0$. Hence $a=c$, and consequently, ax^2+bx+c can be written as $ax^2+bx+a = a(x^2+1)+bx$

This shows that

$$\text{Kernel of } T = \mathrm{Span}\{x^2+1, x\}.$$

On the other hand, x^2+1, x are linearly independent, hence $\{x^2+1, x\}$ is a basis of the kernel of T.

Q5.

(a) (i) $T_A : \mathbb{R}^3 \to \mathbb{R}^4$, defined by $T_A(\vec{x}) = A\vec{x}$, for $\vec{x}$ in $\mathbb{R}^3$. So Kernel of T_A = Nul A.

To find Nul A, we solve the linear system $A\vec{x} = \vec{0}$.

Perform row operations on A:

$$\begin{bmatrix} 5 & -3 & 2 \\ -1 & 1 & 3 \\ 4 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 3 \\ 4 & 1 & -1 \\ 5 & -3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 5 \\ 0 & -7 & -9 \\ 0 & -13 & -8 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 5 \\ 0 & -1 & 1 \\ 0 & -13 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 3 & 5 \\ 0 & -13 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \\ 0 & 0 & -21 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ pivots.

$\Rightarrow$ A has no non-pivot columns, so

Kernel of T_A = Nul A = $\{\vec{0}\}$, ← the zero vector space

and there is no basis of Kernel of T_A.

(ii) Since $\begin{bmatrix} 5 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}$, are all pivot columns, hence they form a basis of Col A.

Q5.

(b) Let c_1, c_2, c_3, c_4 be scalars such that

$$\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} + c_4 \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}.$$

Computing the right-hand side:

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} + c_4 \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & c_1 \\ 0 & 2c_1 \end{bmatrix} + \begin{bmatrix} -c_2 & 2c_2 \\ c_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -c_3 & 2c_3 \end{bmatrix} + \begin{bmatrix} 3c_4 & c_4 \\ 0 & c_4 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 - c_2 + 3c_4 & c_1 + 2c_2 + c_4 \\ c_2 - c_3 & 2c_1 + 2c_3 + c_4 \end{bmatrix}$$

Hence c_1, c_2, c_3, c_4 satisfy the following

$$c_1 - c_2 + 3c_4 = 4$$
$$c_1 + 2c_2 + c_4 = 1$$
$$c_2 - c_3 = 0$$
$$2c_1 + 2c_3 + c_4 = 3.$$

Solving the above system of linear equations, we have

$$c_1 = 2, \quad c_2 = -\frac{5}{7}, \quad c_3 = -\frac{5}{7}, \quad c_4 = \frac{3}{7}.$$

So the coordinates of $\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$ is $(2, -\frac{5}{7}, -\frac{5}{7}, \frac{3}{7})$.

Q6.

(a) Let $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$ in $\mathbb{P}_4$

Note that $f(-t) = a_0 + a_1(-t) + a_2(-t)^2 + a_3(-t)^3 + a_4(-t)^4$

$$= a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4$$

Hence if $f(t)$ is in H, then $f(t) - f(-t) = 0$, which implies

$$2a_1 t + 2a_3 t^3 = 0.$$

Hence $a_1 = a_3 = 0$ in $f(t)$, and $f(t)$ equals

$$a_0 + a_2 t^2 + a_4 t^4.$$

This implies the following description of H:

$$H = \{ a_0 + a_2 t^2 + a_4 t^4 \mid a_0, a_2, a_4 \text{ are in } \mathbb{R} \}$$

$$= \text{Span}\{1, t^2, t^4\}.$$

On the other hand, $1, t^2, t^4$ are linearly indept, hence $\{1, t^2, t^4\}$ is a basis of H.

Since there are three elements in $\{1, t^2, t^3\}$, we see that dimension of $H = 3$.

Q6. (b) Let $B_0 = \{1, t, t^2, t^3, t^4\}$ be the standard basis. Then the B_0-coordinate vector of $t-2$, t^2-t+4, t^2+1 are

$$[t-2]_{B_0} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [t^2-t+4]_{B_0} = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [t^2+1]_{B_0} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

we will show that $[t-2]_{B_0}$, $[t^2-t+4]_{B_0}$ and $[t^2+1]_{B_0}$ are linearly independent, hence $t-2$, t^2-t+4 and t^2+1 are also linearly independent. Consider the corresponding matrix and performing row reduction to an echelon form:

$$\begin{bmatrix} -2 & 4 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 4 & 1 \\ 0 & 1 & 3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 4 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

All three columns are pivot columns, hence

$$[t-2]_{B_0} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [t^2-t+4]_{B_0} = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [t^2+1]_{B_0} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

are linearly independent as claimed.