Math 235 Practice Midterm 1: Solutions

Q1: (a)

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 \quad + 2x_3 &= -2 \\ -2x_1 - x_2 - 3x_3 &= 0 \\ 4x_1 \quad + 4x_3 &= -4 \end{aligned}$$

(b) $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & -2 \\ -2 & -1 & -3 & 0 \\ 4 & 0 & 4 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 2 & -2 \\ 0 & 1 & 1 & 2 \\ -2 & -1 & -3 & 0 \\ 4 & 0 & 4 & -4 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 2 & 0 & 2 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 2 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The reduced echelon form of A.

(c) $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The pivot positions are circled.

(d). The system has infinitely many solutions.

Justification: (i) The last column is non-pivot, so the system is consistent, i.e, has a solution

(ii) There is a free variable, so there must be infinitely many solutions if there is a solution.

Q2: (a). Solve the equation $7x_1+11x_2+13x_3=0$ (for x_1), we have

$$x_1 = -\frac{11}{7}x_2 - \frac{13}{7}x_3.$$

so $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{11}{7}x_2 - \frac{13}{7}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -\frac{11}{7} \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -\frac{13}{7} \\ 0 \\ 1 \end{bmatrix}$$

Hence the plane defined by $7x_1+11x_2+13x_3=0$ is the span of vectors $\vec{v}_1 = \begin{bmatrix} -\frac{11}{7} \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -\frac{13}{7} \\ 0 \\ 1 \end{bmatrix}$

(b) Since A is a 5×3 matrix, it can have at most 3 pivot positions. On the other hand, A has 5 rows, so that one of the rows of A must contain no pivot positions. Hence the equation $A\vec{x}=\vec{b}$ can not be consistent for every vector $\vec{b}$.

Q3. (a) The argumented matrix is $\begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{bmatrix}$.

Transform it to its reduced echelon form:

$$\begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced echelon form

$$\begin{aligned} x_1 + 3x_2 &= -2 \\ x_3 &= 1 \end{aligned} \implies \begin{aligned} x_1 &= -2 - 3x_2 \\ x_2 &= x_2 \\ x_3 &= 1 \end{aligned}$$

So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-3x_2 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \vec{p} + s\vec{v},$$

where $\vec{p} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, s in $\mathbb{R}$.

(b) The general solution for the homogeneous equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\vec{v} = s\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad s \text{ in } \mathbb{R}.$$

Q4: (a) Let $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 4 & 12 & 2 \\ 10 & 7 & -5 \\ 6 & 1 & 3 \end{bmatrix}$

Then $\vec{v}_1, \vec{v}_2\ \vec{v}_3$ are linearly independent if and only if $A\vec{x} = \vec{0}$ has only trivial solution. We determine this by transform A to an echelon form:

$$\begin{bmatrix} 4 & 12 & 2 \\ 10 & 7 & -5 \\ 6 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 1 \\ 10 & 7 & -5 \\ 6 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 1 \\ 0 & -23 & -10 \\ 0 & -17 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & \frac{10}{23} \\ 0 & -17 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & \frac{10}{23} \\ 0 & 0 & \frac{170}{23} \end{bmatrix}$$

Since there are no non-pivot columns in A, $A\vec{x} = \vec{0}$ must have only trivial solution.

Hence $\vec{v}_1, \vec{v}_2\ \vec{v}_3$ are linearly independent.

(b) There is no such example.

<u>The reason</u>: every set of 4 vectors from $\mathbb{R}^3$ are linearly dependent.

(c) <u>Here is a counterexample</u>: 3 distinct vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

There are three subsets of 2 vectors: $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, $\{\vec{v}_2, \vec{v}_3\}$. All of them are linearly dependent.

<u>Q5</u>: (a) The matrix is $A = [T(\vec{e}_1) \ T(\vec{e}_2) \ T(\vec{e}_3)]$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) T is onto if and only if every row of A contains a pivot position.

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last row contains no pivot position. So T is not onto.

(c) T is one to one if and only if A contains no non-pivot columns. From the work in part (b), the last column of A is non-pivot. Hence T is not one to one.

(d) The geometric transformation on $\mathbb{R}^2$ is a composition of the following 3 transformations:

(1) $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ -y \end{bmatrix}$, (2) $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{2}x \\ y \end{bmatrix}$

(3) $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ 3y \end{bmatrix}$

So the transformation is

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ -y \end{bmatrix} \to \begin{bmatrix} \frac{1}{2}x \\ -y \end{bmatrix} \to \begin{bmatrix} \frac{1}{2}x \\ -3y \end{bmatrix}$$

Hence the matrix associated to it is

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -3 \end{bmatrix}.$$

<u>Q6</u>: (a) $U(\vec{x}) = T(S(\vec{x})) = T(B\vec{x}) = A(B\vec{x}) = (AB)\vec{x}$

Hence the standard matrix for U is

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\times0+1\times2 & 0\times1+1\times0, & 0\times2+1\times1 \\ 1\times0+0\times2 & 1\times1+0\times0, & 1\times2+0\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(b) part (i): compute A^2.

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2+2\times5+(-2)\times1 & 2\times2+2\times1+(-2)\times5 & 2\times(-2)+2\times(-3)+(-2)\times(-3) \\ 5\times2+1\times5+(-3)\times1 & 5\times2+1\times1+(-3)\times5 & 5\times(-2)+1\times(-3)+(-3)\times(-3) \\ 1\times2+5\times5+(-3)\times1 & 1\times2+5\times1+(-3)\times5 & 1\times(-2)+5\times(-3)+(-3)\times(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -4 & -4 \\ 12 & -4 & -4 \\ 24 & -8 & -8 \end{bmatrix}$$

<u>part (ii)</u>: $B^T = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}$

$$B^TA = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix} =$$

$$\begin{bmatrix} 1\times2+1\times5+(-1)\times1 & 1\times2+1\times1+(-1)\times5 & 1\times(-2)+1\times(-3)+(-1)\times(-3) \\ (-1)\times2+2\times5+3\times1 & (-1)\times2+2\times1+3\times5 & (-1)\times(-2)+2\times(-3)+3\times(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & -2 \\ 11 & 15 & -13 \end{bmatrix}$$