Math 235 Practice Final Exam Solutions

Q1. (a) First, transform A to an echelon form by row reduction:

$$A=\begin{bmatrix}1&2&1&0&1\\3&6&2&1&0\\2&4&4&-2&8\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&1&0&1\\0&0&-1&1&-3\\0&0&2&-2&6\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&1&0&1\\0&0&-1&1&-3\\0&0&0&0&0\end{bmatrix}$$

Hence $\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}$ are the pivot columns in A, which form a basis of $\mathrm{Col}(A)$. On the other hand, the row vectors in the echelon form, $(1,2,1,0,1)$, $(0,0,-1,1,-3)$ form a basis of $\mathrm{Row}(A)$.

(b) Since $B\vec{x}=\vec{0}$ has 3 free variables, $\mathrm{Nul}(B)$ has dimension 3, i.e, $\dim \mathrm{Nul}(B)=3$. On the other hand, B is a 4×6 matrix, so by the Rank Theorem, $\mathrm{rank}\, B = 6-\dim\mathrm{Nul}(B) = 6-3=3$. Consequently, $\mathrm{Col}\, B$ is 3-dimensional, so it cannot equal $\mathbb{R}^4$.

(c) Since the first two rows are non-zero rows and the last row is zero in an echelon form, C has 2 pivot positions, so that $\mathrm{rank}\, C=2$. Hence $\dim\mathrm{Col}(C)=2$, and by the Rank Theorem, $\dim\mathrm{Nul}(C)=7-\mathrm{rank}\, C=7-2=5$, as C has 7 columns.

Q2. (a) We first find the eigenvalues of $T = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

The characteristic equation is $\det(T - \lambda I_2) = 0$, which is $(-2-\lambda)(-2-\lambda) - 1 = 0$. Solving the equation, we get

$$\lambda = -1, \quad \lambda = -3,$$

which are the eigenvalues of T. Since they are distinct, T is diagonalizable, i.e, $T = PDP^{-1}$, where $D = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$.

(b) To find the matrix P in part (a), we need to find an eigenvector for each of $\lambda = -1$, $\lambda = -3$.

(i) $\lambda = -1$: Solve $(T - (-1) I_2)\vec{v}_1 = \vec{0}$, where $\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

we have $\begin{bmatrix} -2-(-1) & 1 \\ 1 & -2-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(ii) $\lambda = -3$, Similarly, solve $(T - (-3) I_2)\vec{v}_2 = \vec{0}$.

$$\begin{bmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So $P = [\vec{v}_1, \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Q3. (a) $\det(A-\lambda I_3) = \begin{vmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\left((2-\lambda)(2-\lambda)-1\right)$

$= (1-\lambda)(\lambda^2-4\lambda+3) = 0.$

(b) First, find the eigenvalues of A:

$(1-\lambda)(\lambda^2-4\lambda+3) = (1-\lambda)(\lambda-1)(\lambda-3) = 0$

So the eigenvalues of A are $\lambda = 1$, $\lambda = 3$.

(i) The eigenspace of $\lambda = 1$:

$A - 1\cdot I_3 = \begin{bmatrix} 1-1 & 6 & 0 \\ 0 & 2-1 & 1 \\ 0 & 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\xrightarrow{\text{row reduction}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since there is only one non-pivot column, $\dim \mathrm{Nul}(A - 1\cdot I_3) = 1$. Hence the dimension of the eigenspace of $\lambda = 1$ equals 1.

(ii) The eigenspace of $\lambda = 3$:

$A - 3\cdot I_3 = \begin{bmatrix} 1-3 & 6 & 0 \\ 0 & 2-3 & 1 \\ 0 & 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{row reduction}}$

$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which implies $\dim \mathrm{Nul}(A - 3\cdot I_3) = 1$.

Hence the dimension of the eigenspace of $\lambda = 3$ equals 1.

(c) Since A is 3×3 matrix, and the sum of the dimensions of the eigenspaces of $A = 1+1 = 2 \neq 3$, A is NOT diagonalizable.

Q4. (a) $\det(A-\lambda I_2) = \begin{vmatrix} 1-\lambda & -3 \\ 6 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda)+18 = \lambda^2 - 8\lambda + 25 = 0$

We obtain $\lambda = 4+3i$, $\lambda = 4-3i$, which are the eigenvalues of A.

(b) We choose eigenvalue $\lambda = 4-3i$, and find a complex eigenvector $\vec{v}$ of $4-3i$ as follows: Let $\vec{v} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, then

$$\begin{bmatrix} 1-(4-3i) & -3 \\ 6 & 7-(4-3i) \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ which gives } \begin{bmatrix} -3+3i & -3 \\ 6 & 3+3i \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We choose one solution $z_1 = 1+i$, $z_2 = -2$, so that

$$\vec{v} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1+i \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i = \vec{v}_1 + \vec{v}_2 i$$

Let $C = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$, $P = [\vec{v}_1\ \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$. Then $A = PCP^{-1}$.

Q5. We first find $(\mathrm{Col}\,B)^{\perp}$. Because $(\mathrm{Col}\,B)^{\perp} = \mathrm{Nul}(B^T)$, we solve the linear system $B^T\vec{x} = \vec{0}$.

$$B^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ (reduced echelon form of B^T)

The corresponding equations are
$$\begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Consequently,

$(\mathrm{Col}\,B)^{\perp} = \mathrm{Nul}(B^T) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$, and $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ forms a basis of $(\mathrm{Col}\,B)^{\perp}$.

Next, we find $\mathrm{Nul}\,B$. Performing row reductions,

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

from which, we find $\mathrm{Nul}\,B = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. Let $\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in $(\mathrm{Nul}\,B)^{\perp}$. Then $\vec{u} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$, which gives

$$x_1 - x_2 + x_3 = 0$$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

It follows $(\text{Nul } B)^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, and the

vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $(\text{Nul } B)^\perp$.

<u>Q6</u>. (a) We check by a direct calculation that $\vec{u}_i \cdot \vec{u}_j = 0$ for any $i \neq j$, so $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set;

$\vec{u}_1 \cdot \vec{u}_2 = 1 \times 1 + 1 \times (-1) + 0 \times 1 = 0$, $\vec{u}_1 \cdot \vec{u}_3 = 1 \times 1 + 1 \times (-1) + 0 \times (-2) = 0$,

$\vec{u}_2 \cdot \vec{u}_3 = 1 \times 1 + (-1) \times (-1) + 1 \times (-2) = 1 + 1 - 2 = 0$.

On the other hand, $\mathbb{R}^3$ is 3-dimensional, so $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ must be a basis of $\mathbb{R}^3$. Hence $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis.

(b) Since $\vec{u}_1, \vec{u}_2$ are orthogonal, $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis of W. Therefore

$$\text{Proj}_W \vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2, \text{ where}$$

$$c_1 = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{4 \times 1 + 0 \times 1 + 5 \times 0}{1^2 + 1^2 + 0^2} = \frac{4}{2} = 2$$

$$c_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{4 \times 1 + 0 \times (-1) + 5 \times 1}{1^2 + (-1)^2 + 1^2} = \frac{9}{3} = 3,$$

and $\text{Proj}_W \vec{y} = 2\vec{u}_1 + 3\vec{u}_2 = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$.

(c) First, we compute $\text{Proj}_L \vec{y}$:

$$\text{Proj}_L \vec{y} = \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\vec{u}_3$$

$$= \frac{4\times 1 + 0\times(-1) + 5\times(-2)}{1^2+(-1)^2+(-2)^2}\vec{u}_3$$

$$= \frac{-6}{6}\vec{u}_3 = -\vec{u}_3.$$

So the distance between $\vec{y}$ and $L = \text{span}\{\vec{u}_3\}$ is

$\|\vec{y} - \text{Proj}_L\vec{y}\| = \|\vec{y} - (-\vec{u}_3)\| = \|\vec{y} + \vec{u}_3\|$, where

$\vec{y}+\vec{u}_3 = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$. Hence the distance

equals $\sqrt{5^2+(-1)^2+3^2} = \sqrt{35}$.