Math 235 Practice Final Exam

Q1.

(a) Consider the following 3×5 matrix A:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 3 & 6 & 2 & 1 & 0 \\ 2 & 4 & 4 & -2 & 8 \end{bmatrix}$$

Find a basis of Col(A) and a basis of Row(A).

(b) Suppose B is a 4×6 matrix. If the homogeneous equation $B\mathbf{x} = \mathbf{0}$ has 3 free variables, is Col(B) equal to $\mathbb{R}^4$? Justify your answer carefully.

(c) Let C be a 3×7 matrix. Suppose in an echelon form of C, the first two rows are non-zero rows and the last row is a zero row. What are the dimensions of Col(C) and Nul(C)?

Q2. Let $T = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

(a) Find a diagonal matrix D that is similar to T. Justify your answer.

(b) Construct an invertible 2×2 matrix P such that $T = PDP^{-1}$, where D is the matrix from Part (a).

Q3. Let

$$A = \begin{bmatrix} 1 & 6 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Find the characteristic equation of A.

(b) Find the dimension of each eigenspace of A.

(c) Is A diagonalizable? Justify your answer.

Q4. Consider the matrix $A = \begin{bmatrix} 1 & -3 \\ 6 & 7 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find an invertible matrix P such that $A = PCP^{-1}$ for some rotation-scaling matrix C (i.e., $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for some real numbers a, b).

Q5. Let $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. Find a basis of $(\mathrm{Col}\ B)^{\perp}$ and a basis of $(\mathrm{Nul}\ B)^{\perp}$.

Q6. Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix},$$

and let $W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) Show that the collection $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

(b) Find $\mathrm{proj}_W\, \mathbf{y}$, the orthogonal projection of $\mathbf{y}$ onto W.

(c) Compute the distance between $\mathbf{y}$ and $L = \mathrm{Span}\{\mathbf{u}_3\}$.