Towards an equivariant version of Gromov-Taubes invariant

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An overview of the talk:

**Part I:** Brief review of Taubes’ work "Seiberg-Witten=Gromov".

**Part II:** Taubes’ theorems in an equivariant setting: an examination.

**Part III:** Smooth classification of $\mathbb{Z}_n$-Hirzebruch surfaces.
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Part III: Smooth classification of $\mathbb{Z}_n$-Hirzebruch surfaces.
The set-up: Let \((X, \omega)\) be a symplectic 4-manifold with \(b_2^+ > 1\). Fix a \(\omega\)-compatible almost complex structure \(J\), and endow \(X\) with the resulting Riemannian metric \(g(\cdot, \cdot) := \omega(\cdot, J\cdot)\). The \(J\) gives rise to a decomposition \(T^*X \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X\). Denote by \(K_X = \Lambda^2 T^{1,0}X\) the canonical line bundle.

Seiberg-Witten invariant: \(SW_X: \{\text{Spin}^c\text{-structures}\} \to \mathbb{Z}\).

1. The canonical \(\text{Spin}^c\)-structure: \(S_0^+ = I \oplus K_X^{-1}, S_0^- = T^{0,1}X\).

2. All \(\text{Spin}^c\)-structures: \(S^E_+ = E \oplus K_X^{-1} \otimes E, S^E_- = T^{0,1}X \otimes E\), where \(E\) is a complex line bundle.

3. Virtual dimension of the SW moduli space \(M_E\): \(\dim M_E = 2d_E\), where

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d_E = \frac{1}{2}(e \cdot e - c \cdot e), \quad \text{where} \ e = c_1(E), \ c = c_1(K_X).
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The 1-parameter family of SW equations: Let parameter $r \gg 0$, and let $A$ be a $U(1)$-connection on $\text{det} S^E_+$, and $\psi$ be a smooth section of $S^E_+$. The SW equations are the following equations for pairs $(A, \psi)$

$$D_A \psi = 0, F^+_A = q(\psi) + \mu_r$$

where $\mu_r := -\frac{ir}{4} \omega + F^+_{A_0}$. Here $A_0$ is a canonical $U(1)$-connection on $\text{det} S^0_+ = K_X^{-1}$.

Re-formulation: Introduce $(a, (\alpha, \beta))$ by writing $A = A_0 + 2a$ and $\psi = \sqrt{r} \cdot (\alpha, \beta)$, where $a$ is a $U(1)$-connection on $E$, and $\alpha, \beta$ are smooth sections of $E$, $K_X^{-1} \otimes E$ respectively.
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Theorem A (Taubes): Let $(X, \omega)$ be symplectic 4-manifold, $b_2^+ > 1$.

(1) $SW_X(K_X) = \pm 1$.

(2) Suppose $SW_X(E) \neq 0$. Then fix any set $\Omega$ of $d_E$ distinct points, for any parameter $r$ there is a solution $(A, \psi)$ of the $r$-version of SW equations such that $\Omega \subset \alpha^{-1}(0)$. Moreover, as $r \to \infty$, the zero set $\alpha^{-1}(0)$ converges in $C^0$-topology to a finite union of $J$-holomorphic curves $\{C_i\}$, and

$$c_1(E) = \sum_i m_i C_i,$$

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Theorem B (Taubes): Assume $J$ and $\Omega$ are generic. Then

(1) (Regularity) The $J$-holomorphic curves $\{C_i\}$ obey the following constraints:
- Each $C_i$ is embedded and $C_i, C_j$ are disjoint for $i \neq j$.
- The multiplicity $m_i = 1$ unless $C_i$ is a torus with $C_i^2 = 0$.
- There are only finitely many such sets $\{C_i\}$.
- There is a well-defined invariant, denoted by $GT_X(E)$, which is an algebraic count of such sets of $J$-holomorphic curves $\{C_i\}$.

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The set-up: Let \((X, \omega)\) be a symplectic 4-manifold equipped with a symplectic \(G\)-action by a finite group \(G\). Assume \(b_2^+ > 1\), where \(b_2^+ = \dim H^2,+(X; R)^G\). Fix a \(\omega\)-compatible, \(G\)-invariant almost complex structure \(J\), and endow \(X\) with the resulting \(G\)-invariant Riemannian metric \(g(\cdot, \cdot) := \omega(\cdot, J \cdot)\).

Example 1: Holomorphic finite group actions on Kähler surfaces.

Example 2: One can perform an equivariant version of knot surgery on a K3 surface equipped with a finite automorphism group to produce infinitely many distinct symplectic, non-Kähler, homotopy K3 surfaces, each equipped with a symplectic finite group action by a K3 group.

Equivariant SW invariant:

\[ SW_X^G : \{G\text{-complex line bundles}\} \rightarrow \mathbb{Z}. \]
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Theorem B of Taubes in the equivariant setting?

Question: Can one, by choosing generic $G$-invariant $J$, arrange to have the $J$-holomorphic curves $\{C_i\}$ to be embedded and disjoint, and the multiplicities $m_i = 1$ except for $C_i$ being a torus with $C_i^2 = 0$ (of course, $\bigcup_i C_i$ continue to be $G$-invariant)?

Answer: in general, no!.

Example 1: By Theorem A (1), $c_1(|K_X|) = \sum_i m_i C_i$, where $\bigcup_i C_i$ is $G$-invariant. Assume $\{C_i\}$ are embedded and disjoint. Then $m_i > 1$ if $C_i$ contains an isolated fixed point.

Reason: Near $C_i$, $K_X = N_{C_i}^{m_i}$ where $N_{C_i}$ is the normal bundle of $C_i$, which is naturally an equivariant bundle. On the other hand, $K_X \neq N_{C_i}$ as equivariant bundles by examining the weights of the group action at a fixed point.
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Example 2: Let \((X, \omega)\) be a symplectic homotopy K3, and let \(G = \mathbb{Z}_p\) for an odd prime \(p\), which acts symplectically on \((X, \omega)\). Furthermore, assume that the \(G\)-action is homologically trivial. Note that \(b_2^+ G = 3\).

Conjecture: Such group actions do not exist!

Claim: Assume further that the action has only isolated fixed points. Then for \(p > 3\), the \(J\)-holomorphic curves \(\{C_i\}\) in \(c_1(|K_X|) = \sum_i m_i C_i\) in this example can not be made both embedded and disjoint, even for a generic \(G\)-invariant \(J\).
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Explaination: By Theorem A (1), \( c_1(\vert K_X \vert) = \sum_i m_i C_i \), where \( \bigcup_i C_i \) is \( G \)-invariant. By choosing a generic \( G \)-invariant \( J \), one can decompose \( \bigcup_i C_i = \bigcup_j \Lambda_j \), where \( \{ \Lambda_j \} \) are disjoint, such that \( \Lambda_j \) belongs to one of the following cases

- a sphere with one cusp singularity, which is a fixed-point of \( G \), and \( p > 5 \);
- a union of two embedded \((-2)\)-spheres intersecting at one point with tangency of order 2, and \( p > 3 \);
- a union of three embedded \((-2)\)-spheres intersecting at one point transversely, and \( p > 3 \);
- an embedded torus with a non-free \( G \)-action, and \( p = 3 \);
- an embedded torus with a free \( G \)-action.
This motivates the following

**Problem:** Construct Gromov-Taubes invariant of singular spaces (e.g. symplectic 4-orbifolds, normal projective surfaces).

More concretely,

(1) Define a Gromov-Taubes invariant which algebraically counts embedded and disjoint pseudo-holomorphic curves in the complement of the singular set, and

(2) Relate the Gromov-Taubes invariant so defined to the corresponding Seiberg-Witten invariant.
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(2) Relate the Gromov-Taubes invariant so defined to the corresponding Seiberg-Witten invariant.
Consider the following situation: let \((X, \omega)\) be symplectic 4-manifold equipped with a symplectic \(G\)-action, such that \(b_2^+ < 1\). Suppose \(C\) is an embedded, \(G\)-invariant \(J\)-holomorphic curve, such that the induced \(G\)-action on \(C\) is effective. Furthermore, assume either \(C^2 < 0\) or \(C^2 = 0\) and the induced \(G\)-action on \(C\) is not free. Finally, assume that the \(G\)-invariant \(J\) is regular (e.g. \(J\) is a generic \(G\)-invariant almost complex structure).
Consider the following situation: let \((X, \omega)\) be symplectic 4-manifold equipped with a symplectic \(G\)-action, such that \(b_2^+ G > 1\). Suppose \(C\) is an embedded, \(G\)-invariant \(J\)-holomorphic curve, such that the induced \(G\)-action on \(C\) is effective. Furthermore, assume either \(C^2 < 0\) or \(C^2 = 0\) and the induced \(G\)-action on \(C\) is not free. Finally, assume that the \(G\)-invariant \(J\) is regular (e.g. \(J\) is a generic \(G\)-invariant almost complex structure).
Theorem: Assume the above, then

(1) The virtual dimension of the moduli space containing $C$ is 0.

(2) There is a $G$-complex line bundle $E$ canonically associated to $C$ such that $c_1(\|E\|) = C$. Moreover, the virtual dimension of the equivariant SW moduli space associated to $E$ equals 0, and $SW^G_X(E) = \pm 1$.

(3) There is a well-defined invariant, denoted by $GT^G_X(E)$, which is an algebraic count of such embedded, $G$-invariant $J$-holomorphic curves $C$ which determines the same $G$-bundle $E$.

(4) In this case, $SW^G_X(E) = GT^G_X(E)$. 

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Remarks: It is well-known that in the non-equivariant setting, for a generic \( J \) the only \( J \)-holomorphic curves with negative self-intersection are the \((-1)\)-spheres. With this understood, suppose in the above theorem, the \( G \)-invariant \( J \)-holomorphic curve \( C \) has negative self-intersection but is not a \((-1)\)-sphere.

Then observe that

(1) the ordinary SW invariant \( SW_X(|E|) = 0 \), while
(2) the equivariant SW invariant \( SW^G_X(E) \neq 0 \).
Denote by $Z_n$ the cyclic group of order $n$.

**Definition:** A $Z_n$-Hirzebruch surface is a Hirzebruch surface equipped with a holomorphic $Z_n$-action which is homologically trivial.

**Problem:** Classify $Z_n$-Hirzebruch surfaces up to orientation-preserving equivariant diffeomorphisms.
Examples: Fix a generator of $\mathbb{Z}_n$. Then to any pair $(a, b)$ and $r$, where $a, b, r \in \mathbb{Z}$, one can associate a $\mathbb{Z}_n$-Hirzebruch surface

$$F_r(a, b) := P(L_r(a, b) \oplus L_0(a, 0)).$$

Here $L_r(a, b)$ stands for the $\mathbb{Z}_n$-holomorphic line bundle over $CP^1$ of degree $r$ such that

- the action of $\mathbb{Z}_n$ on the base $CP^1$ is given by
  
  $$[z_0: z_1] \rightarrow [\exp\left(\frac{2\pi ia}{n}\right)z_0: z_1],$$

- the action of $\mathbb{Z}_n$ on the fiber at the fixed point $[0: 1]$ is

  $$w \rightarrow \exp\left(-\frac{2\pi ib}{n}\right)w.$$

Canonical equivariant diffeomorphisms: There are 6 types

\[ c_i : F_r(a, b) \to F_{r'}(a', b'), \quad i = 1, 2, \cdots, 6. \]

(1) \( c_1 \) exists if \( a' = -a, \ b' = -b, \) and \( r' = r. \)
(2) \( c_2 \) exists if \( a' = -a, \ b' = b + ra, \) and \( r' = r. \)
(3) \( c_3 \) exists if \( a' = a, \ b' = -b, \) and \( r' = -r. \)
(4) \( c_4 \) exists if \( r' = r = 0, \) and \( a' = b, \ b' = a. \)

In the following two cases, assume \( \gcd(a, n) = \gcd(a', n) = 1. \)
(5) \( c_5 \) exists if \( a' = a, \ b' = b, \) and \( r' = r \mod 2n. \)
(6) \( c_6 \) exists if \( a' = a, \ b' = b, \) and \( r'a' = -2b - ra \mod 2n. \)
Theorem (D. Wilczynski) Any $Z_n$-Hirzebruch surface is holomorphically conjugate to $F_r(a, b)$ for some $r$ and $(a, b)$.

Theorem (Chen, 2015) Two $Z_n$-Hirzebruch surfaces are orientation-preservingly equivariantly diffeomorphic if and only if they can be connected by a finite sequence of canonical equivariant diffeomorphisms.

Remarks: For the pseudo-free case, the result was proved earlier by D. Wilczynski using his topological classification theorem for pseudo-free, locally linear $Z_n$-actions on simply connected 4-manifolds. Our proof is independent, using the equivariant Gromov-Taubes invariant, and it works for non-pseudo-free actions as well.
Smooth classification of $Z_n$-Hirzebruch surfaces

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Key Proposition: Suppose $n$ is even, $0 \leq b' = b \leq \frac{n}{2}$, $0 \leq r' < n$, $b + r' < n$, and $r = r' + n$. There exists no orientation-preserving equivariant diffeomorphisms from $F_r(1, b)$ to $F_{r'}(1, b')$.

Note: $F_r(1, b)$, $F_{r'}(1, b')$ have the same fixed-point set data.

Proof has 2 ingredients: Fix a $\mathbb{Z}_n$-invariant Kähler form $\omega$.

(1) For $F_r(1, b)$, $F_{r'}(1, b')$, if there exists a smoothly embedded $\mathbb{Z}_n$-invariant $(-r')$-sphere, then for any generic $\mathbb{Z}_n$-invariant $J$, there exists a $J$-holomorphic $\mathbb{Z}_n$-invariant $(-r')$-sphere.

(2) For a generic $\mathbb{Z}_n$-invariant $J$, there exists no $J$-holomorphic $\mathbb{Z}_n$-invariant $(-r')$-spheres in $F_r(1, b)$. 
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The end

Thank you!

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