PART 1: ELLIPTIC EQUATIONS

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Contents

1. Elliptic Differential Operators

We present here the basic facts about elliptic differential operators. The treatment follows closely [12].

1.1. **Partial differential operators.** In this section we present some of the basic notions of partial differential operators (p.d.o.) which we need. A basic example of p.d.o. is the *Laplacian*:

$$\Delta : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N) : \Delta u := -\sum_{k=1}^{N} \partial_k^2 u,$$

where $\partial_k := \frac{\partial}{\partial x_k}$, and $x = (x_k)$ is a coordinate system on $\mathbb{R}^N$. The Laplacian Δ is a p.d.o. of order 2. Another example of p.d.o. is the exterior derivative of differential forms

$$d : \Omega^*(\mathbb{R}^N) \to \Omega^{*+1}(\mathbb{R}^N).$$

Let's write out the explicit form for the case of $d : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$. If we identify a 1-form $u = u_1 dx_1 + u_2 dx_2 + u_3 dx_3 \in \Omega^1(\mathbb{R}^3)$ with the column vector $(u_1, u_2, u_3)^T$ and a 2-form $v = v_1 dx_1 \wedge dx_2 + v_2 dx_2 \wedge dx_3 + v_3 dx_1 \wedge dx_3 \in \Omega^2(\mathbb{R}^3)$ with $(v_1, v_2, v_3)^T$, then

$$d : (u_1, u_2, u_3)^T \mapsto (\partial_1 u_2 - \partial_2 u_1, \partial_2 u_3 - \partial_3 u_2, \partial_1 u_3 - \partial_3 u_1)^T,$$

or equivalently,

$$d = A_1 \partial_1 + A_2 \partial_2 + A_3 \partial_3,$$

where A_i, $i = 1, 2, 3$, are the matrices

$$A_1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ A_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ A_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The exterior derivative $d : \Omega^*(\mathbb{R}^N) \to \Omega^{*+1}(\mathbb{R}^N)$ is a p.d.o. of order 1.

In general, a p.d.o. L of order $\leq k$ on $\mathbb{R}^N$, sending a smooth m-vector valued function to a smooth n-vector valued function, has the form

$$L := \sum_{|\alpha| \leq k} A_\alpha(x) \partial^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}^N$, $\alpha_i \geq 0$, $|\alpha| = \sum_{i=1}^N \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$, and $A_\alpha(x)$ are smooth $m \times n$ matrix valued functions on $\mathbb{R}^N$. The operator L is said of order k if $\sum_{|\alpha|=k} A_\alpha(x)\partial^\alpha$ is not identically zero. Note that in the above discussion, $\mathbb{R}^N$ may be replaced by any open subset $D \subset \mathbb{R}^N$.

To define p.d.o. on a smooth manifold, we let M be a smooth manifold (not necessarily compact but with no boundary) of dimension N, and let E, F be smooth (real or complex) vector bundles over M of rank m and n respectively. We denote by $C^\infty(E)$, $C^\infty(F)$ the space of smooth sections of the corresponding bundles, and by $C^\infty(U, E)$, $C^\infty(U, F)$ the subspace of smooth sections whose support lies in a given open subset $U \subset M$.

Definition 1.1. A linear map $L : C^\infty(E) \to C^\infty(F)$ is called a p.d.o. of order k if the following are true:

(1) $\mathrm{supp}(Lu) \subset \mathrm{supp}(u)$, $\forall u \in C^\infty(E)$. Note that this implies $L : C^\infty(U, E) \to C^\infty(U, F)$ for any open subset $U \subset M$;
(2) for any point $p \in M$, there is a smooth chart (U, ϕ) centered at p over which E, F are trivial, such that $L : C^\infty(U, E) \to C^\infty(U, F)$ is given by a p.d.o. of order $\leq k$ on $\phi(U) \subset \mathbb{R}^N$ after fixing a trivialization of E and F over U, and moreover, $L : C^\infty(U, E) \to C^\infty(U, F)$ is of order k for some U.

We remark regarding condition (2) above that the notion of p.d.o. of order k on $\mathbb{R}^N$ is invariant under a coordinate change of $\mathbb{R}^N$. We shall illustrate this for the Laplacian $\Delta = -\sum_{k=1}^N \partial_k^2$ and leave the general case as an exercise.

Let $y = (y_i)$ be another coordinate system on $\mathbb{R}^N$, and set $\partial_i' := \frac{\partial}{\partial y_i}$. Then $\partial_k = \sum_{i=1}^N \frac{\partial y_i}{\partial x_k} \cdot \partial_i'$, and

$$\Delta u = -\sum_{k=1}^N \partial_k^2 u = (\sum_{i,j=1}^N a_{ij}(y)\partial_i'\partial_j' + \sum_{i=1}^N b_i(y)\partial_i')u,$$

where $a_{ij}(y) = -\sum_{k=1}^N \frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k}$ and $b_i(y) = -\sum_{k=1}^N \partial_k(\frac{\partial y_i}{\partial x_k})$. Note that the matrix-valued function $(a_{ij}(y))$ is not identically zero, so that Δ is also of order 2 in the new coordinate system (y_i).

Exercise 1.2. Verify that the notion of a p.d.o. of order k on $\mathbb{R}^N$ is invariant under change of coordinates and trivialization of bundles.

An operator $L : C^\infty(E) \to C^\infty(F)$ is said to be *local* if it has the property $\mathrm{supp}(Lu) \subset \mathrm{supp}(u)$, $\forall u \in C^\infty(E)$. The locality of a p.d.o. on manifold plus the fact that it is given locally by a p.d.o. on $\mathbb{R}^N$ allows us to reduce many aspects in the study of p.d.o. on manifolds to the special case of p.d.o. on $\mathbb{R}^N$ with the aid of a partition of unity. However, the following intrinsic characterization of p.d.o. on manifolds, which is a sort of generalization of the intrinsic characterization of tangent vectors on manifolds, proves to be also very useful.

Let $Op(E,F)$ be the space of linear maps (called operators) $T : C^\infty(E) \to C^\infty(F)$. For any $f \in C^\infty(M)$, we define

$$\mathrm{ad}(f) : Op(E,F) \to Op(E,F), \mathrm{ad}(f)T := [T,f] = T \circ f - f \circ T, \forall T \in Op(E,F).$$

Above, f denotes the $C^\infty(M)$-module multiplication by the function f. With this understood, for any integer $k \geq 0$, we define $PDO^{(k)}(E,F)$ inductively as follows:

$$PDO^{(0)}(E,F) := \cap_{f \in C^\infty(M)} \ker \mathrm{ad}(f),$$

and

$$PDO^{(k)}(E,F) := \{T \in Op(E,F) | \mathrm{ad}(f)T \in PDO^{(k-1)}(E,F), \forall f \in C^\infty(M)\}.$$

Notice that $PDO^{(k)}(E,F) \subset PDO^{(k+1)}(E,F)$, $\forall k \geq 0$. For any $k > 0$, we set $PDO^k(E,F)$ to be the subset of $PDO^{(k)}(E,F)$ which consists of elements not belonging to $PDO^{(k-1)}(E,F)$. Finally, set $PDO(E,F) := \cup_{k \geq 0} PDO^{(k)}(E,F)$.

First, some elementary properties of $PDO^{(k)}(E,F)$ are collected in the following proposition.

Proposition 1.3. *(1) $PDO^{(0)}(E,F)$ may be identified with the space of smooth sections of the vector bundle $Hom(E,F)$ over M. Here $Hom(E,F)$ is the bundle whose fiber at $p \in M$ consists of homomorphisms from E_p into F_p.*

(2) Let $k \geq 0$. Any $L \in PDO^{(k)}(E,F)$ is a local operator, i.e.,

$$supp(Lu) \subset supp(u), \forall u \in C^\infty(E).$$

(3) If $P \in PDO^{(k)}(E,F)$, $Q \in PDO^{(l)}(F,G)$, then $Q \circ P \in PDO^{(k+l)}(E,G)$.

(4) If $L : C^\infty(E) \to C^\infty(F)$ is a p.d.o. of order k, then $L \in PDO^{(k)}(E,F)$.

Proof. (1) Exercise.

(2) We argue by induction on k. For $k = 0$ the claim is obvious. Let $L \in PDO^{(k+1)}(E,F)$, and $u \in C^\infty(E)$. For every $f \in C^\infty(M)$, we have

$$L(fu) = [L,f]u + fLu.$$

Since $[L,f] = \mathrm{ad}(f)L \in PDO^{(k)}(E,F)$, we deduce by induction

$$\mathrm{supp}(L(fu)) \subset \mathrm{supp}(u) \cup \mathrm{supp}(f).$$

Now for any open subset U such that $\mathrm{supp}(u) \subset U$, we pick an $f \in C^\infty(M)$ such that $f \equiv 1$ on $\mathrm{supp}(u)$ and $\mathrm{supp}(f) \subset U$. Then since $fu \equiv u$, we have

$$\mathrm{supp}(Lu) \subset U, \ \forall U \text{ such that } \mathrm{supp}(u) \subset U.$$

It follows that $\mathrm{supp}(Lu) \subset \mathrm{supp}(u)$.

(3) We argue by induction over $k+l$. For $k+l=0$ it is obvious. In general, if $f \in C^\infty(M)$, then

$$[Q \circ P, f] = [Q, f] \circ P + Q \circ [P, f].$$

By induction, the operators on the left-hand side all belong to $PDO^{(k+l-1)}(E,G)$. Hence the claim $Q \circ P \in PDO^{(k+l)}(E,G)$.

(4) Exercise. (Hint: Let $k \geq 1$. Then for any $f \in C^\infty(M)$, $\mathrm{ad}(f)L$ is a p.d.o. of order $k-1$. The claim follows from induction.)

□

Our next goal is to explain the proof of the following theorem. Along the way we shall introduce the important notion of the *principal symbol* of a p.d.o.

Theorem 1.4. *An $L \in Op(E,F)$ is a p.d.o. of order k iff $L \in PDO^k(E,F)$.*

We shall illustrate the proof by first considering the special case of $PDO^1(E,F)$ where E,F are trivial bundles of rank 1. Note that in this case, both $C^\infty(E)$ and $C^\infty(F)$ are naturally identified with $C^\infty(M)$.

Suppose $L \in PDO^1(E,F)$. Then for any $f \in C^\infty(M)$, $\mathrm{ad}(f)L = [L,f] \in PDO^{(0)}(E,F)$, hence there exists a $\sigma(f) \in C^\infty(M)$ such that $[L,f]u = \sigma(f)u$, $\forall u \in C^\infty(E)$. One can easily check that for any $f, g \in C^\infty(M)$,

$$\sigma(fg) = \sigma(f)g + f\sigma(g).$$

By the intrinsic characterization of tangent vecters on manifolds, there exists a unique smooth vector field X on M such that

$$\sigma(f) = Xf, \ \ \forall f \in C^\infty(M).$$

We define $L_0 \in Op(E,F)$ by $L_0 u := Xu$. Then it is easy to check that L_0 is a p.d.o. of order 1. (Note that here the vector field X is not identically zero because $L \in PDO^1(E,F)$.) Moreover, for any $f \in C^\infty(M)$ $\mathrm{ad}(f)L_0 = Xf$, hence

$$\mathrm{ad}(f)(L - L_0) = 0, \ \ \forall f \in C^\infty(M).$$

This implies that $L - L_0 \in PDO^{(0)}(E,F)$, and $L = L_0 + a$ for some $a \in C^\infty(M)$. It follows that L is a p.d.o. of order 1.

Let's consider more generally, for any $k \geq 1$, $L \in PDO^{(k)}(E,F)$. We introduce

$$\sigma(L)(f_1, \cdots, f_k) := \frac{1}{k!}\mathrm{ad}(f_1)\cdots \mathrm{ad}(f_k)L \in PDO^{(0)}(E,F), \forall f_i \in C^\infty(M).$$

Lemma 1.5. *(1) $\sigma(L)(f_1, \cdots, f_k)$ is symmetric in $f_1, \cdots, f_k$.*

(2) For any $p \in M$, $\sigma(L)(f_1, \cdots, f_k)(p)$ depends only on the values $df_1(p), \cdots, df_k(p)$.

Proof. (1) Exercise. (Hint: Jacobi identity plus $[f,g]=0$, $\forall f, g \in C^\infty(M)$.)

(2) By the multi-linearality of $\sigma(L)(f_1, \cdots, f_k)$ in $f_1, \cdots, f_k$ and the symmetric property in part (1), it suffices to show that if $df_1(p) = 0$, then $\sigma(L)(f_1, \cdots, f_k)(p) = 0$.

Set $Q := \frac{1}{k!}\mathrm{ad}(f_2)\cdots \mathrm{ad}(f_k)L \in PDO^{(1)}(E,F)$. (When $k=1$, $Q = L$.) We need to show that $\mathrm{ad}(f_1)Q(p) = 0$. For this we recall the fact that $df_1(p) = 0$ implies that there exist smooth functions α_j, β_j (at least locally which suffices) which vanish at p such that

$$f_1 = f_1(p) + \sum_j \alpha_j \beta_j.$$

Now we have

$$\mathrm{ad}(f_1)Q = [Q, \sum_j \alpha_j \beta_j] = \sum_j ([Q, \alpha_j]\beta_j + \alpha_j [Q, \beta_j]),$$

which gives $\mathrm{ad}(f_1)Q(p) = 0$, for $\alpha_j(p) = \beta_j(p) = 0$ and $[Q, \alpha_j], [Q, \beta_j] \in PDO^{(0)}(E, F)$. □

Let $L \in PDO^{(k)}(E, F)$. For any $p \in M$, Lemma 1.5 gives rise to a symmetric k-multilinear map

$$\sigma(L)(p) : T_p^* M \times \cdots \times T_p^* M \to \mathrm{Hom}(E_p, F_p).$$

It is a standard algebraic fact that $\sigma(L)(p)$ as a symmetric, k-multilinear map is determined by the corresponding homogeneous polynomial of degree k in ξ:

$$\sigma(L)(p)(\xi) := \sigma(L)(p)(\xi, \cdots, \xi), \quad \xi \in T_p^* M.$$

In this way $\{\sigma(L)(p) : p \in M\}$ defines a smooth section $\sigma(L)$ of the vector bundle $\mathrm{Hom}(\pi^* E, \pi^* F)$ over $T^* M$, where $\pi^* E, \pi^* F$ are the pull-back bundles of E, F via $\pi : T^* M \to M$. When $L \in PDO^{(0)}(E, F)$, we define $\sigma(L) := \pi^* L$. Here L is regarded as a smooth section of $\mathrm{Hom}(E, F)$ over M, and $\pi^* L$ is the pull-back section of the pull-back bundle $\mathrm{Hom}(\pi^* E, \pi^* F)$ over $T^* M$.

Definition 1.6. For any $L \in PDO^k(E, F)$, $\sigma(L)$ is called the *principal symbol* of L.

Exercise 1.7. (1) Show that for any $L \in PDO^{(k)}(E, F)$, $\sigma(L)$ is not identically zero iff $L \in PDO^k(E, F)$.

(2) Consider the special case $M = \mathbb{R}^N$, where E, F are necessarily trivial bundles. Let L be a p.d.o. of order k on $\mathbb{R}^N$, given by

$$L = \sum_{|\alpha| \le k} A_\alpha(x) \partial^\alpha.$$

Note that by Proposition 1.3(4), $L \in PDO^{(k)}(E, F)$. Show that

$$\sigma(L)(x)(\xi) = \sum_{|\alpha| = k} A_\alpha(x) \xi^\alpha, \forall x \in \mathbb{R}^N$$

where $\xi = \xi_1 dx_1 + \cdots + \xi_N dx_N$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N}$ for $\alpha = (\alpha_1, \cdots, \alpha_N)$.

(3) Prove Theorem 1.4, by modifying the proof for the special case where $k = 1$ and E, F are trivial bundles of rank 1.

(4) Show that $\sigma(Q \circ P) = \sigma(Q) \circ \sigma(P)$.

Definition 1.8. (1) A p.d.o. $L : C^\infty(E) \to C^\infty(F)$ on M is said to be *elliptic* if for any $p \in M$ and $\xi \in T_p^* M \setminus \{0\}$,

$$\sigma(L)(p)(\xi) : E_p \to F_p$$

is an isomorphism.

(2) More generally, a complex of p.d.o.

$$0 \;\rightarrow\; C^\infty(E_0) \;\stackrel{L_0}{\rightarrow}\; C^\infty(E_1) \;\stackrel{L_1}{\rightarrow}\; C^\infty(E_2) \;\rightarrow\; \cdots \;\rightarrow C^\infty(E_n) \;\rightarrow\; 0$$

(where $L_{i+1} \circ L_i = 0, \forall i \geq 0$) is called an *elliptic complex* if for any $p \in M$ and $\xi \in T_p^* M \setminus \{0\}$, the associated complex of principal symbols

$$0 \;\longrightarrow\; E_0|_p \;\stackrel{\sigma(L_0)(p)(\xi)}{\longrightarrow}\; E_1|_p \;\stackrel{\sigma(L_1)(p)(\xi)}{\longrightarrow}\; E_2|_p \;\longrightarrow\; \cdots \;\longrightarrow E_n|_p \;\longrightarrow\; 0$$

is exact.

Observe that if $P \in PDO^k(E,F), Q \in PDO^l(F,G)$ are elliptic, then $Q \circ P \in PDO^{k+l}(E,G)$ and is also elliptic.

Next we discuss the notion of *formal adjoint* of a p.d.o. To this end we need further assume M is oriented and is endowed with a Riemannian metric g and the bundles E, F are endowed with a metric $\langle\cdot,\cdot\rangle_E$, $\langle\cdot,\cdot\rangle_F$ respectively (in the case of complex vector bundles, endowed with a Hermitian metric). Finally, we denote the corresponding space of compactly supported smooth sections by $C_0^\infty(E)$, $C_0^\infty(F)$. We should point out that the notion of formal adjoint depends on the choice of the additional data g, $\langle\cdot,\cdot\rangle_E$, and $\langle\cdot,\cdot\rangle_F$.

Definition 1.9. Let $P \in PDO(E,F)$. An operator $Q \in Op(F,E)$ is said to be a formal adjoint of P if $\forall u \in C_0^\infty(E), v \in C_0^\infty(F)$, we have

$$\int_M \langle Pu, v\rangle_F \, dVol_g = \int_M \langle u, Qv\rangle_E \, dVol_g.$$

It turns out that every $P \in PDO(E,F)$ admits a unique formal adjoint, which will be denoted by P^*.

Exercise 1.10. (1) Show that formal adjoints (assuming they exist) are unique.

(2) Show that (a) for $L_1, L_2 \in PDO(E,F)$, if L_1^*, L_2^* exist, then $(L_1+L_2)^*$ exists and $(L_1+L_2)^* = L_1^* + L_2^*$, (b) for $P \in PDO(E,F)$, $Q \in PDO(F,G)$, if P^*, Q^* exist, then $(Q\circ P)^*$ exists, and $(Q \circ P)^* = P^* \circ Q^*$.

(3) If $L \in PDO^k(E,F)$, then $L^* \in PDO^k(F,E)$. Moreover,

$$\sigma(L^*) = (-1)^k \sigma(L)^*.$$

Here $\sigma(L)^*$ denotes the transpose (conjugate transpose in the Hermitian case) of $\sigma(L)$ as a linear map. In particular, the formal adjoint of an elliptic p.d.o is again an elliptic p.d.o of the same order.

Definition 1.11. A p.d.o. $L \in PDO(E,E)$ is said to be *formally self-adjoint* if $L = L^*$.

Proposition 1.12. *Every $P \in PDO(E,F)$ admits at least one formal adjoint.*

Proof. (*A sketch.*) Suppose $\{\phi_i\}$ is a partition of unity on M. Then $P = \sum_{i,j} P_{ij}$, where $P_{i,j} := \phi_i \circ P \circ \phi_j \in PDO(E,F)$. Furthermore, if each $P_{i,j}^*$ exists, so does P^*. This allows to reduce the problem to the special case of $M = \mathbb{R}^N$ by picking a partition of unity such that each $P_{i,j}$ is given by a p.d.o. on $\mathbb{R}^N$.

□

Exercise 1.13. (1) Prove the existence of formal adjoint for the case of a p.d.o. on $\mathbb{R}^N$. (Note that the Riemannian metric on $\mathbb{R}^N$ is not necessarily Euclidean.)

(2) Give the details of the proof of Proposition 1.12.

We end this section by discussing some clasical examples of p.d.o.-s on manifolds.

Example 1.14. (1) Let (M,g) be an oriented Riemannian manifold of dimension n. Consider the Laplacian

$$\Delta : C^\infty(M) \to C^\infty(M), \Delta u := -*d*du, \forall u \in C^\infty(M),$$

where $* : \Lambda^*(M) \to \Lambda^{n-*}(M)$ is the Hodge $*$-operator, which is characterized by $\alpha \wedge *\beta = \langle \alpha, \beta \rangle_g dVol_g$. In a local coordinate system (x_i),

$$\Delta u = -\sum_{i,j=1}^{n} (g^{ij}\partial_i\partial_j u + \frac{1}{\sqrt{det(g_{ij})}}\partial_i(\sqrt{det(g_{ij})}g^{ij})\partial_j u),$$

where $\partial_i := \frac{\partial}{\partial x_i}$, $g_{ij} = \langle \partial_i, \partial_j \rangle_g$, and $(g^{ij}) = (g_{ij})^{-1}$. (The Laplacian $\Delta := -\sum_{k=1}^N \partial_k^2$ on $\mathbb{R}^N$ is a special case with $M = \mathbb{R}^N$ endowed with the Euclidean metric.) From the local description above, it follows that Δ is a second order p.d.o. on M. Its principal symbol is given by

$$\sigma(\Delta)(p)(\xi) = -\sum_{i,j=1}^{n} g^{ij}(p)\xi_i\xi_j = -|\xi|_g^2, \ \ \forall p \in M,$$

where $\xi = \sum_{i=1}^n \xi_i dx_i|_p \in T_p^*M$. In particular, Δ is an elliptic p.d.o. Finally, we observe that Δ is formally self-adjoint: $\forall u, v \in C_0^\infty(M)$,

$$\int_M (\Delta u)v dVol_g = \int_M -(d*du)v = \int_M dv \wedge *du = \int_M du \wedge *dv = \int_M u(\Delta v)dVol_g.$$

(2) Let E be a smooth vector bundle over a smooth manifold M. A connection on E (also called a covariant derivative) is a linear map

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E),$$

which satisfies: $\forall f \in C^\infty(M), u \in C^\infty(E)$,

$$\nabla(fu) = df \otimes u + f\nabla u.$$

Note that ∇ is a p.d.o. of order 1, i.e., $\nabla \in PDO^1(E, T^*M \otimes E)$, because $\forall f \in C^\infty(M)$ $\mathrm{ad}(f)\nabla := [\nabla, f] = df\otimes \in PDO^{(0)}(E, T^*M \otimes E)$. The principal symbol is given by

$$\sigma(\nabla)(p)(\xi) = \xi\otimes, \ \ \forall p \in M, \xi \in T_p^*M.$$

Suppose furthermore, M is oriented and endowed with a Riemannian metric g, E is endowed with a metric $\langle \cdot, \cdot \rangle$. We determine the formal adjoint ∇^* of ∇ in local coordinates. Suppose (e_i) is a positively oriented local orthonormal frame of T^*M, (α_j) is a local orthonormal frame of E, and (∂_i) is the local frame of TM dual to (e_i). Moreover, suppose

$$\nabla \alpha_k = \sum_{i,j} \Gamma_k^{ij} e_i \otimes \alpha_j, \ \ de_k = \sum_{s,t} \omega_k^{st} e_s \wedge e_t.$$

Then

$$\nabla^*(\sum_{i,j} v_{ij} e_i \otimes \alpha_j) = \sum_k (\sum_i -\partial_i v_{ik} + \sum_{i,j} \Gamma_k^{ij} v_{ij})\alpha_k.$$

(Here one uses the fact that $\omega_j^{ji} = 0, \forall i, j$. As an exercise verify the above formula for ∇^*!) From the local description we can read off its principal symbol

$$\sigma(\nabla^*)(p)(\xi) : (e_i \otimes \alpha_j)|_p \mapsto -\xi_i \alpha_j|_p, \forall p \in M,$$

where $\xi = \sum_i \xi_i e_i|_p \in T_p^* M$. Observe that

$$\sigma(\nabla^*) = -\sigma(\nabla)^*,$$

which is a special case of the general fact

$$\sigma(L^*) = (-1)^k \sigma(L)^*, \forall L \in PDO^k(E, F).$$

The associated covariant Laplacian is the second order p.d.o

$$\Delta = \Delta_\nabla : C^\infty(E) \to C^\infty(E), \Delta := \nabla^* \nabla.$$

It is an elliptic p.d.o because its principal symbol

$$\sigma(\Delta)(p)(\xi) = \sigma(\nabla^*) \circ \sigma(\nabla) = -|\xi|_g^2, \ \forall p \in M, \xi \in T_p^* M.$$

(3) Let M be a smooth n-manifold. The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a first order p.d.o. For any $f \in C^\infty(M)$, $\omega \in \Omega^k(M)$,

$$(\mathrm{ad}(f)d)\omega = d(f\omega) - f d\omega = df \wedge \omega.$$

Hence the principal symbol of d is given by

$$\sigma(d)(p)(\xi) = e(\xi), \forall p \in M, \xi \in T_p^* M,$$

where $e(\xi)$ denotes the exterior multiplication by ξ. From this it is easy to check that the deRham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots \to \Omega^n(M) \to 0$$

is an elliptic complex, i.e., the associated complex of principal symbols

$$0 \to \Lambda_p^0(M) \xrightarrow{e(\xi)} \Lambda_p^1(M) \xrightarrow{e(\xi)} \Lambda_p^2(M) \longrightarrow \cdots \longrightarrow \Lambda_p^n(M) \longrightarrow 0$$

is exact for any $p \in M$ and $\xi \in T_p^* M \setminus \{0\}$.

Assume M is oriented and endowed with a Riemannian metric g. Let $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ be the formal adjoint of d. One easily finds that $d^* = (-1)^{nk+n+1} * d *$, where $*$ is the Hodge $*$-operator.

The Hodge-deRham operator is the formally self-adjoint, first order p.d.o.

$$\delta := d + d^* : \Omega^*(M) \to \Omega^*(M).$$

We claim it is an elliptic operator. To see this, note that its principal symbol $\sigma(\delta)(p)(\xi) = e(\xi) - e(\xi)^*$, and it suffices to show that $\sigma(\delta)(p)(\xi)$ has trivial kernel for any $\xi \neq 0$.

Suppose $(e(\xi) - e(\xi)^*)\alpha = 0$. Then $e(\xi)^* \circ e(\xi)\alpha = 0$ (since $e(\xi)^* \circ e(\xi)^* = 0$), which implies that

$$\langle e(\xi)\alpha, e(\xi)\alpha \rangle_g = \langle \alpha, e(\xi)^* \circ e(\xi)\alpha \rangle_g = 0.$$

Hence $e(\xi)\alpha = 0$. Since the deRham complex is elliptic, there exists a β such that $\alpha = e(\xi)\beta$. Then $e(\xi)^*\alpha = 0$ implies $e(\xi)^* \circ e(\xi)\beta = 0$, which implies similarly $\alpha = e(\xi)\beta = 0$. Hence $\sigma(\delta)(p)(\xi)$ is invertible for any $\xi \neq 0$, and the Hodge-deRham operator δ is elliptic. (Note that the ellipticity of δ follows formally from the ellipticity of the corresponding deRham complex.)

Exercise 1.15. (1) For any $\xi \in T_p^*M$, let $\xi^* \in T_pM$ be the metric dual of ξ, and let $i(\xi^*)$ denote the interior multiplication by ξ^*. Show that $e(\xi)^* = i(\xi^*)$.

(2) Let $\Delta := \delta^2 = dd^* + d^*d$ be the Hodge Laplacian. Then Δ is a formally self-adjoint, second order elliptic p.d.o. Show that $\sigma(\Delta)(p)(\xi) = -|\xi|_g^2$, $\forall p \in M$, $\xi \in T_p^*M$.

Definition 1.16. Let E be a smooth vector bundle over a Riemannian manifold (M, g). A second order p.d.o.

$$L : C^\infty(E) \to C^\infty(E)$$

is called a *generalized Laplacian* if $\sigma(L)(p)(\xi) = -|\xi|_g^2$ for any $p \in M$, $\xi \in T_p^*M$.

Exercise 1.17. Suppose (M, g) is oriented. Prove the following fact: for any formally self-adjoint, generalized Laplacian $L : C^\infty(E) \to C^\infty(E)$ (with respect to some metric on E), there exists a unique connection ∇ on E compatible with the metric such that

$$L = \nabla^*\nabla + R,$$

where R is a smooth section of $\mathrm{End}(E)$ over M (or equivalently, $R \in PDO^{(0)}(E, E)$). In favorable situations where R is "positive", since $\nabla^*\nabla$ is always semi-positive, this allows to prove that the equation $Lu = 0$ has only trivial solution. For geometrically defined generalized Laplacian L, R is often expressed in terms of various curvatures (the so-called Bochner technique or Weitzenböck formula).

As an example, let (M, g) be a compact, oriented Riemannian manifold, and let ∇ be the Levi-Civita connection. Then the Hodge Laplacian $\Delta := dd^* + d^*d : \Omega^1(M) \to \Omega^1(M)$ satisfies

$$\Delta = \nabla^*\nabla + \mathrm{Ric},$$

where Ric is the Ricci tensor, which is regarded as a self-adjoint endomorphism of T^*M via the metric duality. On the other hand, the kernel of Δ is the space of harmonic 1-forms on M, which via the Hodge theory identifies with the first deRham cohomology group $H^1_{dR}(M)$. Here is the upshot: if (M, g) has positive Ricci curvature, then $H^1_{dR}(M) = 0$. (Proof: if $\Delta u = 0$, then

$$0 = \int_M \langle (\nabla^*\nabla + \mathrm{Ric})u, u\rangle_g dVol_g = \int_M (|\nabla u|_g^2 + \langle \mathrm{Ric}\ u, u\rangle_g) dVol_g,$$

which implies that $u \equiv 0$ because $\langle \mathrm{Ric}\ u, u\rangle_g \geq 0$ and $\langle \mathrm{Ric}\ u, u\rangle_g \equiv 0$ iff $u \equiv 0$.)

Example 1.14-cont. (Cauchy-Riemann operator.) Let M be a complex manifold of (complex) dimension n and E a holomorphic vector bundle over M. The Dolbeault operator $\bar\partial : \Omega^{0,*}(E) \to \Omega^{0,*+1}(E)$ is the first order p.d.o. which is locally defined as follows: let (z^i) be a local holomorphic coordinate system on M and let (s^α) be a holomorphic local frame of E, then

$$\bar{\partial}(\sum_{A,\alpha} f_{A,\alpha} d\bar{z}^A \otimes s^\alpha) = \sum_{i,A,\alpha} \frac{\partial f_{A,\alpha}}{\partial \bar{z}^i} d\bar{z}^i \wedge d\bar{z}^A \otimes s^\alpha.$$

(Here $\Omega^{0,*}(E) := C^\infty(\Lambda^{0,*}(M) \otimes E)$, and A denotes a multi-index of non-negative integers.) To determine the principal symbol, note that $\forall f \in C^\infty(M)$,

$$\mathrm{ad}(f)\bar{\partial} := [\bar{\partial}, f] = (\sum_i \frac{\partial f}{\partial \bar{z}^i} d\bar{z}^i)\wedge = e((df)^{0,1}),$$

where $(df)^{0,1}$ is the $(0,1)$-component of df viewed as a section of the complexified cotangent bundle, and $e(\cdot)$ denotes the exterior multiplication. It follows that

$$\sigma(\bar{\partial})(p)(\xi) = e(\xi^{0,1}), \forall p \in M, \xi \in T_p^* M,$$

where $\xi^{0,1}$ denotes the $(0,1)$-component of ξ viewed as an element of $T_p^*M \otimes \mathbb{C}$. As in the case of deRham complex, it follows similarly from the above description of $\sigma(\bar{\partial})(p)(\xi)$ that the Dolbeault complex

$$0 \rightarrow \Omega^{0,0}(E) \xrightarrow{\bar{\partial}} \Omega^{0,1}(E) \xrightarrow{\bar{\partial}} \Omega^{0,2}(E) \rightarrow \cdots \rightarrow \Omega^{0,n}(E) \rightarrow 0$$

is also an elliptic complex. In the case when $n = 1$, e.g., $M = \Sigma$ is a Riemann surface, the Dolbeault complex reduces to a first order elliptic p.d.o.

$$\bar{\partial} : C^\infty(E) \rightarrow C^\infty(\Lambda^{0,1} \otimes E),$$

which is called a *Cauchy-Riemann operator*. Note that its principal symbol is given by the exterior multiplication $e(\xi^{0,1})$.

Definition 1.18. Let E be a complex vector bundle over a Riemann surface Σ. A first order p.d.o.

$$L : C^\infty(E) \rightarrow C^\infty(\Lambda^{0,1} \otimes E)$$

is called a *generalized Cauchy-Riemann operator* if $\sigma(L)(p)(\xi) = e(\xi^{0,1})$ for any $p \in \Sigma$, $\xi \in T_p^*\Sigma$. In particular, a generalized Cauchy-Riemann operator is elliptic.

Exercise 1.19. Let L be a generalized Cauchy-Riemann operator. Show that $2L^*L$ is a generalized Laplacian.

1.2. **Sobolev spaces and Hölder spaces.** In this section we present the necessary tools from functional analysis in the study of p.d.o.-s. Throughout, for $1 \leq p < \infty$ and $D \subset \mathbb{R}^N$ an open set, $L^p(D)$ will denote the classical Banach space of p-integrable Lebesgue measurable functions over D with the norm

$$||u||_p = ||u||_{p,D} := (\int_D |u|^p dx)^{1/p}, \forall u \in L^p(D).$$

When $p = 2$, $L^2(D)$ is a Hilbert space with $\langle u, v\rangle = \int_D uv\, dx$, $\forall u, v \in L^2(D)$. Finally,

$$L^p_{loc}(D) := \{u | \phi u \in L^p(D), \forall \phi \in C_0^\infty(D)\} \text{ (called locally } L^p\text{-functions).}$$

The following inequality (called Hölder's inequality) will be frequently used:

$$||uv||_r \leq ||u||_p \cdot ||v||_q, \text{ where } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Exercise 1.20. Use Hölder's inequality in the following:

(1) Show that $||u||_q \leq ||u||_p^\lambda ||u||_r^{1-\lambda}$ where $p \leq q \leq r$ and $1/q = \lambda/p + (1-\lambda)/r$.

(2) Prove that $\forall \epsilon > 0$, $||u||_q \leq \epsilon ||u||_r + \epsilon^{-\mu}||u||_p$ where $\mu = (1/p - 1/q)/(1/q - 1/r)$. (Hint: use $|ab| \leq |a|^p/p + |b|^q/q$ for any $a, b \in \mathbb{R}$ where $1 = 1/p + 1/q$.)

(3) Derive Young's inequality:

$$||u * v||_p \leq ||u||_1 \cdot ||v||_p, \ \ \forall u \in L^1(\mathbb{R}^N), v \in L^p(\mathbb{R}^N),$$

where $u * v$ is the convolution

$$u * v(x) = \int_{\mathbb{R}^N} u(x-y)v(y)dy, \forall x \in \mathbb{R}^N.$$

(4) Show that $L^p_{loc}(D) \subset L^1_{loc}(D)$ for $p \geq 1$.

Definition 1.21. (1) Let $u, v \in L^1_{loc}(D)$. We say $\partial_k u = v$ *weakly* if

$$\int_D v\phi \, dx = -\int_D u \partial_k \phi \, dx, \ \ \forall \phi \in C_0^\infty(D).$$

(Note that if $\partial_k u = v_i$ weakly, $i = 1, 2$, then $v_1 = v_2$ *a.e.* in D.) Moreover, v is called the *weak ∂_k-derivative* of u and we say $\partial_k u$ exists weekly. (As an exercise, check that if $u \in C^1(D)$, $v \in C^0(D)$, then $\partial_k u = v$ weakly iff $\partial_k u = v$ classically.)

(2) More generally, for any p.d.o. $L : C^\infty(D) \to C^\infty(D)$ (including as a special case higher order partial derivatives $L = \partial^\alpha$ where α is a multi-index), $Lu = v$ *weakly* if

$$\int_D v\phi \, dx = \int_D uL^*\phi \, dx, \ \ \forall \phi \in C_0^\infty(D).$$

(Here the formal adjoint L^* is with respect to the Euclidean metric on D.)

Definition 1.22. (Sobolev spaces.) Let $k > 0$ be an integer, $1 \leq p < \infty$. Set

$$L^{k,p}(D) := \{u \in L^p(D) | \partial^\alpha u \text{ exists weakly and } \partial^\alpha u \in L^p(D), \forall \alpha, |\alpha| \leq k\}.$$

The functions in $L^{k,p}(D)$ are called $L^{k,p}$-functions on D, which come with a natural norm

$$||u||_{k,p} = ||u||_{k,p,D} := (\sum_{|\alpha| \leq k} ||\partial^\alpha u||_p^p)^{1/p}, \ \ \forall u \in L^{k,p}(D).$$

The locally $L^{k,p}$-functions on D are defined and denoted by

$$L^{k,p}_{loc}(D) := \{u | \phi u \in L^{k,p}(D), \forall \phi \in C_0^\infty(D)\}.$$

Lemma 1.23. *Let $(u_n) \subset L^{k,p}(D)$ be a sequence. If there exists a $v_0 \in L^p(D)$, and for any multi-index α, $0 < |\alpha| \leq k$, there exists a $v_\alpha \in L^p(D)$, such that*

$$u_n \to v_0, \ \ \partial^\alpha u_n \to v_\alpha \ \ in \ L^p(D), \ \ as \ n \to \infty,$$

then $\partial^\alpha v_0 = v_\alpha$ weakly and $\lim_{n\to\infty} u_n = v_0$ in $L^{k,p}(D)$.

Proof. It suffices to show $\partial^\alpha v_0 = v_\alpha$ weakly. For this observe that $\forall \phi \in C_0^\infty(D)$,

$$\begin{aligned}\int_D \partial^\alpha v_0 \, \phi \, dx &= (-1)^{|\alpha|} \int_D v_0 \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \lim_{n\to\infty} \int_D u_n \partial^\alpha \phi \, dx \\ &= \lim_{n\to\infty} \int_D \partial^\alpha u_n \, \phi \, dx = \int_D v_\alpha \phi \, dx.\end{aligned}$$

Above, $\lim_{n\to\infty} \int_D \partial^\alpha u_n \, \phi \, dx = \int_D v_\alpha \phi \, dx$ and $\lim_{n\to\infty} \int_D u_n \partial^\alpha \phi \, dx = \int_D v_0 \partial^\alpha \phi \, dx$ follows from Hölder's inequality and the fact that $\phi, \partial^\alpha \phi \in L^q(D)$ where $\frac{1}{p} + \frac{1}{q} = 1$. ☐

As a corollary, we obtain

Proposition 1.24. *For $1 \leq p < \infty$, $L^{k,p}(D)$ is a Banach space which is reflexive if $1 < p < \infty$.*

(Recall a Banach space V is called *reflexive* if $(V^*)^* = V$, where V^* denotes the dual space of V (the space of functionals on V). For $1 < p < \infty$, $L^p(D)$ is reflexive with dual space $L^q(D)$, $\frac{1}{p} + \frac{1}{q} = 1$.)

Proof. By Lemma 1.23, the embedding

$$T : L^{k,p}(D) \to L^p(D) \times \cdots \times L^p(D), \;\; T(u) = (\partial^\alpha u | 0 \leq |\alpha| \leq k)$$

has a closed image. The proposition follows from the fact that a closed subspace of a Banach space is a Banach space and furthermore, a closed subspace of a reflexive Banach space is reflexive. ☐

Remark 1.25. (1) When $p = 2$, $L^{k,2}(D)$ is in fact a Hilbert space.

(2) For alternative notations for $L^{k,p}(D)$, some authors use $W^{k,p}(D)$, and $H^k(D)$ for the case of $p = 2$.

An important fact is that $L^{k,p}$-functions can be locally approximated by smooth functions in $L^{k,p}$-norms. To make this precise we will need *mollifiers*. Pick a bump-function $\rho \in C_0^\infty(\mathbb{R}^N)$ such that

$$\rho \geq 0, \;\; \mathrm{supp}(\rho) \subset \{|x| < 1\}, \text{ and } \int_{\mathbb{R}^N} \rho \, dx = 1.$$

Then for each $\delta > 0$, $\delta \to 0$, we define $\rho_\delta(x) := \delta^{-N} \rho(x/\delta)$. Note that

$$\mathrm{supp}(\rho_\delta) \subset \{|x| < \delta\}, \text{ and } \int_{\mathbb{R}^N} \rho_\delta \, dx = 1.$$

The sequence (ρ_δ) is called a *mollifying sequence*. Now let $D_1, D_2 \subset \mathbb{R}^N$ be bounded domains such that $\overline{D_1} \subset D_2$ and $\mathrm{dist}(\partial \overline{D_1}, \partial \overline{D_2}) := \delta_0 > 0$. We assume $\delta < \delta_0$.

Exercise 1.26. Prove the following statements.

(1) For any $u \in L^{k,p}(D_2)$ such that $\mathrm{supp}(u) \subset D_1$, $\rho_\delta * u \in C_0^\infty(D_2)$.

(2) For any multi-index α, $|\alpha| \leq k$, $\partial^\alpha(\rho_\delta * u) = \rho_\delta * \partial^\alpha u$ in $C^\infty(D_2)$.

Lemma 1.27. *For any $u \in L^{k,p}(D_2)$, supp$(u) \subset D_1$, $\rho_\delta * u \to u$ in $L^{k,p}(D_2)$ as $\delta \to 0$.*

Proof. Let α be any multi-index, $0 \leq |\alpha| \leq k$. $\forall \epsilon > 0$, we approximate $\partial^\alpha u$ by a continuous function v_α such that $\text{supp}(v_\alpha) \subset D_2$ and $||v_\alpha - \partial^\alpha u||_p < \epsilon/3$. Note that for sufficiently small $\delta > 0$ (once v_α is chosen), $\rho_\delta * v_\alpha \in C_0^\infty(D_2)$. By Young's inequality, we have

$$||\rho_\delta * v_\alpha - \rho_\delta * \partial^\alpha u||_p \leq ||\rho_\delta||_1 \cdot ||v_\alpha - \partial^\alpha u||_p = ||v_\alpha - \partial^\alpha u||_p < \epsilon/3$$

because $||\rho_\delta||_1 = \int_{\mathbb{R}^R} \rho_\delta \; dx = 1$. It remains to estimate $||\rho_\delta * v_\alpha - v_\alpha||_p$. For any $x \in D_2$,

$$\begin{aligned} |\rho_\delta * v_\alpha(x) - v_\alpha(x)| &= |\int_{D_2} \rho_\delta(x-y)(v_\alpha(y) - v_\alpha(x)) \; dy| \\ &\leq \int_{\{|z|<\delta\}} \rho_\delta(z)|v_\alpha(x-z) - v_\alpha(x))| \; dz \\ &\leq \sup_{|z|<\delta, x\in D_2} |v_\alpha(x-z) - v_\alpha(x))|. \end{aligned}$$

By the uniform continuity of v_α,

$$\sup_{|z|<\delta, x\in D_2} |v_\alpha(x-z) - v_\alpha(x))| < (Volume(D_2))^{-1/p} \cdot (\epsilon/3)$$

when $\delta > 0$ is sufficiently small, which gives $||\rho_\delta * v_\alpha - v_\alpha||_p < \epsilon/3$. Putting the three estimates together,

$$||\partial^\alpha(\rho_\delta * u) - \partial^\alpha u||_p = ||\rho_\delta * \partial^\alpha u - \partial^\alpha u||_p < \epsilon, \; \forall \alpha, |\alpha| \leq k,$$

hence $\rho_\delta * u \to u$ in $L^{k,p}(D_2)$ as $\delta \to 0$. □

Exercise 1.28. Show that $C_0^\infty(\mathbb{R}^N)$ is dense in $L^{k,p}(\mathbb{R}^N)$.

Hint: For any $R > 0$, pick a cut-off function η_R on $\mathbb{R}^N$ such that $\eta_R \equiv 1$ on $\{|x| \leq R\}$ and $\eta_R \equiv 0$ on $\{|x| \geq R+1\}$, and furthermore, $|d\eta_R| \leq 2$. For any multi-index α,

$$\int_{\mathbb{R}^N} |\partial^\alpha(\eta_R u) - \partial^\alpha u|^p \; dx \leq \sum_{|\beta| \leq |\alpha|} \int_{\{|x|\geq R\}} |\partial^\beta u|^p \; dx.$$

Apply Lemma 1.27 to $\eta_R u$.

Remark 1.29. In general, $C_0^\infty(D)$ is not dense in $L^{k,p}(D)$. We denote by $L_0^{k,p}(D)$ the closure of $C_0^\infty(D)$ in $L^{k,p}(D)$. When D is a bounded domain in $\mathbb{R}^N$ with "good" boundary regularity (e.g. $\partial\overline{D} \subset \mathbb{R}^N$ is an embedded submanifold of codimension 1), one can show that $C^\infty(\overline{D})$ is dense in $L^{k,p}(D)$. The idea is for any $u \in L^{k,p}(D)$, we first approximate u in $L^{k,p}(D)$-norm by a $\tilde{u} \in L^{k,p}(\tilde{D})$ with compact support for some $\tilde{D}$ containing $\overline{D}$. Then apply Lemma 1.27 to $\tilde{u}$. See [5].

Exercise 1.30. Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be a C^k-diffeomorphism and $f \in C^k(\mathbb{R}^N)$. For any $u \in L^{k,p}(\mathbb{R}^N)$ with compact support, show that $f \cdot u \circ \phi \in L^{k,p}(\mathbb{R}^N)$. Moreover, the chain rule holds true for the weak partial derivatives of $f \cdot u \circ \phi$ up to order k.

Hint: (1) Let $D \subset \mathbb{R}^N$ be a bounded domain. $\forall v_1, v_2 \in C_0^\infty(D)$,

$$||f \cdot v_1 \circ \phi - f \cdot v_2 \circ \phi||_{k,p} \leq c(f, \phi, D)||v_1 - v_2||_{k,p}.$$

(2) Apply Lemma 1.27 to u. Show that (with help of (1) above) $\{f \cdot (\rho_\delta * u) \circ \phi\}$ converges in $L^{k,p}(\mathbb{R}^N)$ as $\delta \to 0$.

(3) Prove that $f \cdot (\rho_\delta * u) \circ \phi$ converges to $f \cdot u \circ \phi$ in $L^p(\mathbb{R}^N)$.

Note that the above result allows one to define the notion of $L^{k,p}$-sections of a smooth vector bundle over a smooth manifold.

Exercise 1.31. Let $u \in L^p(\mathbb{R}^N)$. Show that the following statements are equivalent.

(1) $u \in L^{1,p}(\mathbb{R}^N)$.

(2) There exists a constant $C > 0$ such that, $\forall \phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$|\int_{\mathbb{R}^N} u\partial_k \phi \, dx| \leq C||\phi||_q, \forall k = 1, \cdots, N$$

where $q = p/(p-1)$.

(3) There exists a constant $C > 0$ such that, for all $h \in \mathbb{R}^N$, we have

$$||\Delta_h u||_p \leq C|h|,$$

where $\Delta_h u(x) := u(x+h) - u(x)$, $\forall x \in \mathbb{R}^N$.

Exercise 1.32. Let $f \in C^1(\mathbb{R})$ such that $|f'| \leq const$. Show that for any $u \in L^{1,p}(\mathbb{R}^N)$ with $f(u) \in L^p(\mathbb{R}^N)$, one has $f(u) \in L^{1,p}(\mathbb{R}^N)$ and $\partial_k \, f(u) = f'(u)\partial_k u$.

Exercise 1.33. Let $u \in L^{1,p}(\mathbb{R}^N)$. Show that $|u| \in L^{1,p}(\mathbb{R}^N)$, and

$$\partial_k |u| = \begin{cases} \partial_k u \;\; a.e. \text{ on } \{u > 0\} \\ 0 \;\; a.e. \text{ on } \{u = 0\} \\ -\partial_k u \;\; a.e. \text{ on } \{u < 0\}. \end{cases}$$

Hint: Show that $u_\epsilon := (\epsilon^2 + u^2)^{1/2}$ converges to $|u|$ in $L^{1,p}(\mathbb{R}^N)$ as $\epsilon \to 0$.

Next we discuss embedding theorems of Sobolev spaces. To this end each space $L^{k,p}(\mathbb{R}^N)$, where $1 \leq p < \infty$, is associated with a "strength"

$$\sigma(k,p) := \sigma_N(k,p) = k - N/p.$$

The geometric meaning of $\sigma(k,p)$ is given in

Exercise 1.34. Let $\lambda > 0$. For any $f \in C_0^\infty(\mathbb{R}^N)$, define $f_\lambda(x) := f(\lambda x)$, $\forall x \in \mathbb{R}^N$. Show that $||\partial^\alpha f_\lambda||_p = \lambda^{\sigma(k,p)}||\partial^\alpha f||_p$ for any α with $|\alpha| = k$.

Theorem 1.35. *(Sobolev) If $\sigma_N(k,p) = \sigma_N(m,q) < 0$ and $k > m$, then there exists a constant $C = C(N,k,m,p,q) > 0$ such that*

$$||u||_{m,q} \leq C||u||_{k,p}, \forall u \in L^{k,p}(\mathbb{R}^N).$$

In particular, there is a continuous inclusion $L^{k,p}(\mathbb{R}^N) \hookrightarrow L^{m,q}(\mathbb{R}^N)$.

The following estimate plays a crucial role.

Lemma 1.36. *Let $N \geq 2$. Then $||u||_{N/(N-1)} \leq (\prod_{i=1}^N ||\partial_i u||_1)^{1/N}$, $\forall u \in L^{1,1}(\mathbb{R}^N)$.*

Proof. The proof relies on the following elementary, but ingenious inequality.

Exercise 1.37. (Gagliardo-Nirenberg) Let $N \geq 2$ and $f_1, \cdots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. For each $x = (x^1, x^2, \cdots, x^N) \in \mathbb{R}^N$ and $1 \leq i \leq N$, define

$$\xi_i := (x^1, \cdots, \hat{x}^i, \cdots, x^N) \in \mathbb{R}^{N-1}.$$

Prove that

$$f(x) := f_1(\xi_1) f_2(\xi_2) \cdots f_N(\xi_N) \in L^1(\mathbb{R}^N),$$

and moreover, $||f||_1 \leq \prod_{i=1}^N ||f_i||_{N-1}$.

Back to the proof of Lemma 1.36. First, assume $u \in C_0^\infty(\mathbb{R}^N)$. We define, for $1 \leq i \leq N$, $g_i := \int_{-\infty}^\infty |\partial_i u| dx^i \in C_0^\infty(\mathbb{R}^{N-1})$ and $f_i := g_i^{1/(N-1)}$. Then

$$|u(x)| = |\int_{-\infty}^{x^i} \partial_i u \, dx^i| \leq g_i(\xi_i),$$

and $|u(x)|^{N/(N-1)} \leq f_1(\xi_1) f_2(\xi_2) \cdots f_N(\xi_N)$. By Gagliardo-Nirenberg,

$$||u||_{N/(N-1)} \leq (\prod_{i=1}^N ||f_i||_{N-1})^{(N-1)/N} = \prod_{i=1}^N ||g_i||_1)^{1/N} = (\prod_{i=1}^N ||\partial_i u||_1)^{1/N}.$$

For arbitrary $u \in L^{1,1}(\mathbb{R}^N)$, use density of $C_0^\infty(\mathbb{R}^N)$ in $L^{1,1}(\mathbb{R}^N)$. □

The Sobolev theorem can be easily reduced to the case of $k = 1$, $m = 0$, which follows from the following estimate.

Lemma 1.38. *Let $N \geq 2$. There exists $C(N, p)$ such that for any $u \in C_0^\infty(\mathbb{R}^N)$,*

$$||u||_{p^*} \leq C(N,p)(\sum_{i=1}^N ||\partial_i u||_p), \text{ where } p^* := Np/(N-p).$$

Proof. Recall the classical arithmetic-geometric means inequality:

$$(a_1 a_2 \cdots a_N)^{1/N} \leq \frac{1}{N}(a_1 + a_2 + \cdots + a_N), \forall a_i \in \mathbb{R}, a_i \geq 0.$$

The case of $p = 1$ follows from Lemma 1.36.

For $p > 1$. We consider $v := |u|^r$ for some $r > 1$ to be determined.

Exercise 1.39. Let $r > 1$ and $u \in C_0^\infty(\mathbb{R}^N)$. Show that $v := |u|^r \in L^{1,1}(\mathbb{R}^N)$ and $\forall i$, $\partial_i v = r|u|^{r-1}\partial_i |u|$. Moreover, show that $|\partial_i |u|| = |\partial_i u|$, $\forall i$.

Now apply Lemma 1.36 to v, and set $q := p/(p-1)$, we obtain (using Hölder)

$$\begin{aligned}
(\int_{\mathbb{R}^N} |u|^{rN/(N-1)} dx)^{(N-1)/N} &\leq (\prod_{i=1}^N \int_{\mathbb{R}^N} r|u|^{r-1}|\partial_i u| dx)^{1/N} \\
&\leq r(\prod_{i=1}^N (\int_{\mathbb{R}^N} |u|^{(r-1)q} dx)^{1/q} \cdot (\int_{\mathbb{R}^N} |\partial_i u|^p dx)^{1/p})^{1/N} \\
&= r(\int_{\mathbb{R}^N} |u|^{(r-1)q} dx)^{1/q} \cdot (\prod_{i=1}^N (\int_{\mathbb{R}^N} |\partial_i u|^p dx)^{1/p})^{1/N}.
\end{aligned}$$

If we choose r such that $rN/(N-1)=(r-1)q$, which means $r=p(N-1)/(N-p)>1$, then $rN/(N-1)=pN/(N-p)=p^*$ and $(N-1)/N-1/q=1/p^*$, and we obtain

$$||u||_{p^*}\leq r(\prod_{i=1}^{N}||\partial_i u||_p)^{1/N}\leq \frac{p(N-1)}{N(N-p)}(\sum_{i=1}^{N}||\partial_i u||_p).$$

□

Proof of Theorem 1.35. Note that if the theorem holds for $k=1$, $m=0$, then it holds for $m=k-1$, $\forall k>0$. Now for any (k,p), (m,q) with $\sigma(k,p)=\sigma(m,q)$, $k>m$, there are $p_1,p_2,\cdots$, such that

$$\sigma(k,p)=\sigma(k-1,p_1)=\sigma(k-2,p_2)=\cdots=\sigma(m,q).$$

The theorem follows easily.

□

Remark 1.40. It is clear from the proof that Theorem 1.35 holds true for $L_0^{k,p}(D)$ for any open subset of $\mathbb{R}^N$. It also holds true for $L^{k,p}(D)$ when D has good boundary regularity, cf. [5].

Theorem 1.41. *(Rellich-Kondrachov) Suppose* $0>\sigma_N(k,p)>\sigma_N(m,q)$ *and* $k>m$. *Then any bounded sequence* $(u_n)\subset L^{k,p}(\mathbb{R}^N)$ *supported in a ball* $B_R(0)$ *of radius* R *has a subsequence which is convergent in* $L^{m,q}(\mathbb{R}^N)$.

Proof. (A sketch.) This precompactness result comes from the following well-known Arzéla-Ascoli theorem on equicontinuous families of functions on bounded domains:

Let $\{u_\alpha\in C^0(D)|\alpha\in A\}$ *be a family of continuous functions on a bounded domain* $D\subset\mathbb{R}^N$ *with the following property: there exists a constant* $C>0$ *such that (1)* $|u_\alpha(x)|\leq C,\forall\alpha\in A, x\in D$, *(2) given any* $\epsilon>0$, *there is a* $\delta>0$ *such that if* $|x-y|<\delta$, *then* $|u_\alpha(x)-u_\alpha(y)|<\epsilon$ *for* $\forall\alpha\in A, x,y\in D$. *Then* $\{u_\alpha\in C^0(D)|\alpha\in A\}$ *is precompact in* $C^0(D)$.

With this understood, the proof consists of two steps: (i) apply the Arzéla-Ascoli theorem to mollifiers $\{\rho_\delta * u_n\}$ for each fixed $\delta>0$, (2) establish convergence $\rho_\delta * u_n\rightarrow u_n$ as $\delta\rightarrow 0$, which is *uniform* in u_n (compare Lemma 1.23). It suffices to consider the case where $k=1$, $m=0$, and $q<p^*:=pN/(N-p)$.

More concretely,

(1) For each $\delta>0$, there exists $C(\delta)>0$ such that

$$|\rho_\delta * u(x)|\leq C(\delta)||u||_p,\;\; |\rho_\delta * u(x)-\rho_\delta * u(y)|\leq C(\delta)||u||_p\cdot|x-y|,\forall x,y\in D.$$

(2) Establish the following estimate: there is a constant $C>0$ such that

$$||\rho_\delta * u-u||_1\leq\delta\cdot C\cdot||u||_{1,p}.$$

(3) Use $||u||_q\leq||u||_1^\lambda||u||_{p*}^{1-\lambda}$ (Exercise 1.20(1)) and then the embedding $L^{1,p}\rightarrow L^{p^*}$.

□

Exercise 1.42. Work out the details of the proof of Theorem 1.41.

Next we define the Hölder spaces $C^{k,\alpha}(D)$, where $D \subset \mathbb{R}^N$ is an open subset, $k \geq 0$, and $0 < \alpha < 1$. Denote by $B_z(R)$ the open ball of radius $R > 0$ centered at $z \in D$. We introduce for any $u \in C^0(D)$

$$\text{osc } u_{z,R} := \sup_{x,y \in B_z(R) \cap D} |u(x) - u(y)|,$$

and define for any $\rho > 0$,

$$[u]_{\alpha,D,\rho} := \sup_{0<R\leq\rho, z\in D} R^{-\alpha}\text{osc } u_{z,R}.$$

Finally, set $||u||_{\infty,D} := \sup_{x\in D} |u(x)|$.

Exercise 1.43. Let $\rho_1 < \rho_2$. Show that $[u]_{\alpha,D,\rho_1} \leq [u]_{\alpha,D,\rho_2}$ and

$$[u]_{\alpha,D,\rho_2} \leq [u]_{\alpha,D,\rho_1} + 2\rho_1^{-\alpha}||u||_{\infty,D}.$$

The above result shows that the following subspace of $C^0(D)$ is independent of the choice of $\rho > 0$:

$$C^{0,\alpha}(D) := \{u \in C^0(D) | ||u||_{\infty,D} + [u]_{\alpha,D,\rho} < \infty\}.$$

We fix a choice of $\rho = 1$, define a norm

$$||u||_{0,\alpha} = ||u||_{0,\alpha,D} = ||u||_{\infty,D} + [u]_{\alpha,D,1}.$$

Finally, $C^{k,\alpha}(D) := \{u \in C^k(D) | \partial^A u \in C^{0,\alpha}(D), \forall 0 \leq |A| \leq k\}$, with

$$||u||_{k,\alpha} = \sum_{0\leq|A|\leq k} ||\partial^A u||_{0,\alpha}, \forall u \in C^{k,\alpha}(D),$$

and $C^{k,\alpha}_{loc}(D) := \{u \in C^k(D) | \phi u \in C^{k,\alpha}(D), \forall \phi \in C_0^\infty(D)\}$.

Proposition 1.44. *(1) The spaces $C^{k,\alpha}(D)$ with norm $|| \cdot ||_{k,\alpha}$ are Banach spaces.*
(2) There are natural inclusions: for $\alpha \geq \beta$, $k \geq 0$, $C^{k,\alpha}(D) \subset C^{k,\beta}(D)$.

Exercise 1.45. (1) Prove Proposition 1.44.
(2) Extend the notion of $C^{k,\alpha}_{loc}$-functions to $C^{k,\alpha}_{loc}$-maps between smooth manifolds.

Theorem 1.46. *(Morrey) If $\sigma(m,p) = \sigma(k,\alpha) := k + \alpha > 0$ and $m > k$, then $L^{m,p}(\mathbb{R}^N)$ embeds continuously in $C^{k,\alpha}(\mathbb{R}^N)$ via inclusion.*

Proof. It suffices to prove the case where $k = 0, m = 1$ and $\alpha = 1 - N/p > 0$. For any $u \in C_0^\infty(\mathbb{R}^n)$, We introduce the average of u over the ball $B_z(R)$:

$$\bar{u}_{z,R} := \frac{1}{Vol(B_z(R))} \int_{B_z(R)} u(y)dy.$$

Then the theorem follows from the following key estimate: $\forall x \in B_z(R)$,

$$|u(x) - \bar{u}_{z,R}| \leq C(N) \int_{B_x(2R)} \frac{|du(y)|}{|x-y|^{N-1}} dy.$$

Here du denotes the gradient vector of u and $|du(y)| = (\sum_{i=1}^N |\partial_i u(y)|^2)^{1/2}$.

To prove the estimate, for any unit vector $\omega \in \mathbb{R}^n$, let $D_\omega u(y)$ be the directional derivative of u at y in the direction of ω. Then

$$\begin{aligned} |u(x) - \bar{u}_{z,R}| &= |\frac{1}{Vol(B_z(R))}\int_{B_z(R)} (u(x)-u(y))dy| \\ &\leq \frac{1}{Vol(B_z(R))}\int_{B_z(R)}\int_0^{|x-y|} |D_\omega(x+t\omega)|dtdy \text{ (where } \omega = \frac{y-x}{|x-y|}) \\ &\leq \frac{1}{Vol(B_z(R))}\int_0^{2R} dt \int_{B_x(2R)} |du(x+t\omega)|dy. \end{aligned}$$

We write y in spherical coordinates (r,ω). Then since $|du(x+t\omega)|$ is constant in r, we have

$$\int_{B_x(2R)} |du(x+t\omega)|dy = \frac{(2R)^N}{N}\int_{S^{N-1}(1)} |du(x+t\omega)|d\omega.$$

This gives, for an appropriate constant $C(N) > 0$,

$$|u(x) - \bar{u}_{z,R}| \leq C(N)\int_0^{2R} dt \int_{S^{N-1}(1)} |du(x+t\omega)|d\omega \leq C(N)\int_{B_x(2R)} \frac{|du(y)|}{|x-y|^{N-1}}dy.$$

With this estimate in hand, we now observe that for $q = p/(p-1)$,

$$\int_{B_x(2R)} \frac{1}{|x-y|^{q(N-1)}}dy = C(N,p)(2R)^{\alpha q}$$

for some $C(N,p) > 0$. Use Hölder inequality, we easily obtain

$$||u||_{0,\alpha} \leq C||u||_{1,p}$$

from which the theorem follow. (As an exercise, work out the details!) □

Remark 1.47. Theorem 1.46 clearly holds true for $L_0^{m,p}(D)$ where D is any open subset of $\mathbb{R}^N$. When D has good boundary regularity, it also holds for $L^{k,p}(D)$, cf. [5].

With Arzéla-Ascoli theorem, we obtain

Corollary 1.48. *If $\sigma(m,p) > \sigma(k,\alpha) := k+\alpha > 0$ and $m > k$, then a bounded sequence $(u_n) \subset L^{m,p}(\mathbb{R}^N)$ with support contained in a fixed ball is precompact in $C^{k,\alpha}(\mathbb{R}^N)$.*

There are interpolation inequalities which allow one to "absorb" terms involving lower order partial derivatives in various estimates. We give a version for Sobolev spaces below. There is a similar one for Hölder spaces, cf. [5].

Theorem 1.49. *Let $D \subset \mathbb{R}^N$ be an open subset and $k \geq 2$. Then there exists $C(k,N) > 0$ such that for any $\epsilon > 0$,*

$$||\partial^\beta u||_p \leq \epsilon||u||_{k,p} + C(k,N)\epsilon^{|\beta|/(|\beta|-k)}||u||_p, \ \ \forall u \in L_0^{k,p}(D),$$

where β is any multi-index with $0 < |\beta| < k$. When D is a bounded domain with good boundary regularity, the above inequality holds for any $u \in L^{k,p}(D)$.

The following exercises are designed to give you some ideas for the interpolation inequalities.

Exercise 1.50. Let $u \in C_0^2(\mathbb{R})$. Then

(1) Show that for any interval (a,b) with $b-a=\epsilon$,

$$|u'(x)| \leq \frac{3}{\epsilon}(|u(x_1)|+|u(x_2)|)+\int_a^b |u^{(2)}|, \forall x \in (a,b), x_1 \in (a, a+\frac{\epsilon}{3}), x_2 \in (b-\frac{\epsilon}{3}, b).$$

With this show that $|u'(x)| \leq \int_a^b |u^{(2)}| + \frac{18}{\epsilon^2}\int_a^b |u|$, and then use Hölder to show

$$\int_a^b |u'|^p \leq 2^{p-1}(\epsilon^p \int_a^b |u^{(2)}|^p + (\frac{18}{\epsilon})^p \int_a^b |u|^p).$$

Sum over intervals of length $\epsilon > 0$, one obtains, for any $\epsilon > 0$,

$$||u'||_p \leq \epsilon ||u^{(2)}||_p + \frac{36}{\epsilon}||u||_p.$$

(2) Use $\int |u'|^2 = -\int u^{(2)}u$ and Hölder inequality to prove a version of $p=2$:

$$||u'||_2 \leq \epsilon ||u^{(2)}||_2 + \frac{1}{4\epsilon}||u||_2.$$

Exercise 1.51. Let D be a bounded domain. Use the fact that for $k > m$, $L_0^{k,p}(D) \to L_0^{m,p}(D)$ sends bounded sets to precompact sets to show that for any $\epsilon > 0$, there exists $C(\epsilon) > 0$, such that

$$||u||_{m,p} \leq \epsilon ||u||_{k,p} + C(\epsilon)||u||_p, \text{ for } k > m, \forall u \in L_0^{k,p}(D).$$

Finally, we discuss Sobolev spaces and Hölder spaces of sections of a smooth vector bundle over a *compact* smooth manifold.

Let E be a smooth vector bundle over a compact oriented smooth manifold M. Fix a Riemannian metric g on M, a metric h on E, and a metric compatible connection (i.e. covariant derivative) ∇ on E, $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$. Then for each $m \geq 0$, there are metrics $\langle \cdot, \cdot \rangle_{(g,h)}$ and metric compatible connections on $(T^*M)^{\otimes m} \otimes E$ induced from g, h and the Levi-Civita connection. Via iteration we obtain higher order covariant derivatives for all orders $m \geq 0$, with $\nabla^0 = Id$ and $\nabla^1 = \nabla$:

$$\nabla^m : C^\infty(E) \to C^\infty((T^*M)^{\otimes m} \otimes E).$$

Given any $k \geq 0$, $1 \leq p < \infty$, we define $\forall u \in C^\infty(E)$,

$$||u||_{k,p}(g,h,\nabla) := (\sum_{m \leq k} \int_M |\nabla^m u|^p_{(g,h)} \, dVol_g)^{1/p}.$$

We define the Sobolev space $L^{k,p}(E)(g,h,\nabla)$ to be the completion of $C^\infty(E)$ under the norm $||\cdot||_{k,p}(g,h,\nabla)$. Then $L^{k,p}(E)(g,h,\nabla)$ is a Banach space, which is reflexive if $p > 1$.

Exercise 1.52. (1) Show that $L^{k,p}(E)(g,h,\nabla)$ consists of $L^{k,p}$-sections of E, i.e., those sections of E which in local trivializations of E can be expressed as n-vector valued functions (here $n = \text{rank } E$) whose components are $L^{k,p}$-functions. In particular, as a set $L^{k,p}(E)(g,h,\nabla)$ does not depend on the data (g,h,∇). We shall denote it by $L^{k,p}(E)$ accordingly.

(2) For any two different choices of data (g_i,h_i,∇_i), $i=1,2$, the norms $||\cdot||_{k,p}(g_i,h_i,\nabla_i)$ are equivalent. We shall fix a choice of (g,h,∇) and denote the norm by $||\cdot||_{k,p}$.

To define Hölder spaces, let $\rho_0>0$ be the injective radius of (M,g). For any $0<\rho\leq\rho_0$, $z\in M$, let $B_z(\rho)$ be the geodesic ball of radius ρ centered at z. Then any $x,y\in B_z(\rho)$, $x\neq y$, is connected by a unique geodesic $\gamma_{x,y}\subset B_z(\rho)$. We denote by $T_{x,y}:(T^*M)^{\otimes m}\otimes E|_x\rightarrow (T^*M)^{\otimes m}\otimes E|_y$, $m\geq 0$, the parallel transport along $\gamma_{x,y}$. ($T_{x,y}$ is determined by (g,h,∇).)

Let $k\geq 0$, $0<\alpha<1$. We define $\forall u\in C^0(E)$

$$||u||_{0,\alpha}:=\sup_{z\in M}|u(z)|+\sup_{0<\rho\leq\rho_0,z\in M}\rho^{-\alpha}\text{osc }u_{z,\rho},$$

where osc $u_{z,\rho}:=\sup_{x,y\in B_z(\rho),x\neq y}|u(y)-T_{x,y}u(x)|$, and $\forall u\in C^k(E)$, $||u||_{k,\alpha}:=\sum_{m\leq k}||\nabla^m u||_{0,\alpha}$. Finally, we set $C^{k,\alpha}(E):=\{u\in C^k(E)|||u||_{k,\alpha}<\infty\}$.

Exercise 1.53. (1) Show that $C^{k,\alpha}(E)$ are independent of the choice of (g,h,∇), and different choices of (g,h,∇) give equivalent norms $||\cdot||_{k,\alpha}$.

(2) Show that $C^{k,\alpha}(E)$ are Banach spaces.

Theorem 1.54. *(1) If $0>\sigma(k,p)\geq\sigma(m,q)$, $k>m$, then there is continuous embedding $L^{k,p}(E)\rightarrow L^{m,q}(E)$ induced via the inclusion map. Moreover, when $\sigma(k,p)>\sigma(m,q)$, any bounded sequence $(u_n)\subset L^{k,p}(E)$ is precompact in $L^{m,q}(E)$.*

(2) If $\sigma(m,p)\geq\sigma(k,\alpha):=k+\alpha$, $m>k$, then there is continuous embedding $L^{m,p}(E)\rightarrow C^{k,\alpha}(E)$ induced via the inclusion map. Moreover, when $\sigma(m,p)>\sigma(k,\alpha)$, any bounded sequence $(u_n)\subset L^{m,p}(E)$ is precompact in $C^{k,\alpha}(E)$.

Exercise 1.55. Prove Theorem 1.54. (Hint: use partition of unity to reduce to the case of $\mathbb{R}^N$.)

1.3. **Apriori estimates and elliptic regularity.** In this section we discuss the interior estimates for elliptic p.d.o-s and regularity of weak and strong solutions. Since these are local issues we shall confine ourselves to the case of $M=D\subset\mathbb{R}^N$ an open subset and E,F are trivial bundles of rank n over D. We further assume, for simplicity, that D, E,F are given standard metrics.

Throughout we let $L:C^\infty(E)\rightarrow C^\infty(F)$ be an elliptic p.d.o. of order $k>0$.

Theorem 1.56. *(Interior Elliptic Estimates) Let D' be any bounded domain such that $\overline{D}'\subset D$. Then*

(1) For any $1<p<\infty$ and integer $m\geq 0$, there exists a constant $C>0$ such that

$$||u||_{m+k,p,D'}\leq C(||Lu||_{m,p,D}+||u||_{p,D}),\ \ \forall u\in C^\infty(E),$$

where C depends on L,m,p,D',D, but is independent of u.

(2) For any $0<\alpha<1$ and integer $m\geq 0$, there exists a constant $C>0$ such that

$$||u||_{m+k,\alpha,D'}\leq C(||Lu||_{m,\alpha,D}+||u||_{0,\alpha,D}),\ \ \forall u\in C^\infty(E).$$

Here $C=C(L,m,\alpha,D',D)$.

We shall explain the proof by breaking it down into several steps. During the course of the proof we will explain in more concrete terms as how the constant C on the right-hand side of the estimates may depend on L. We first recall the relevant results from analysis which we quote without giving proofs.

Denote by $\mathbb{S}^{N-1}$ the unit sphere in $\mathbb{R}^N$. Let $\Omega \in C^\infty(\mathbb{S}^{N-1})$ such that $\int_{\mathbb{S}^{N-1}} \Omega dx = 0$. To each such a function Ω, one can associate an operator T_Ω as follows:

$$T_\Omega u(x) := \lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} \frac{\Omega(y/|y|)}{|y|^N} u(x-y)dy, \ \ \forall u \in C_0^\infty(\mathbb{R}^N).$$

Theorem 1.57. *(1) (Calderon-Zygmund inequality) For $1 < p < \infty$, T_Ω extends to a bounded operator from $L^p(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$: there exists $A(p, N, \Omega) > 0$ such that*

$$||T_\Omega u||_p \leq A(p, \Omega)||u||_p, \ \ \forall u \in L^p(\mathbb{R}^N).$$

(2) (Hölder-Korn-Lichtenstein-Girand) Let $0 < \alpha < 1$. There exists $A(\alpha, R, N, \Omega) > 0$ such that for any $u \in C^\infty(\mathbb{R}^N)$ satisfying $u(x) = 0$ for $|x| \geq R$,

$$||T_\Omega u||_{0,\alpha} \leq A(\alpha, R, \Omega)||u||_{0,\alpha}.$$

Theorem 1.57(2) is elementary and we leave it as an exercise (for a proof see [2]). The Calderon-Zygmund inequality is more involved and we refer to [6], Theorem 4.2.10.

Recall the Fourier transform of a function $u \in C_0^\infty(\mathbb{R}^N)$ is defined by

$$\hat{u}(\xi) := \int_{\mathbb{R}^N} e^{-2\pi i \, \xi \cdot x} u(x) \, dx, \ \ \xi \in \mathbb{R}^N,$$

where $\xi \cdot x = \xi^1 x^1 + \cdots + \xi^N x^N$. The inverse Fourier transform is given by

$$u(x) = \int_{\mathbb{R}^N} e^{2\pi i \, \xi \cdot x} \hat{u}(\xi) \, d\xi, \ \ x \in \mathbb{R}^N.$$

The proof of the following result can be found in [6], Prop. 2.4.7.

Proposition 1.58. *Let $\bar{m} \in C^\infty(\mathbb{S}^{N-1})$ with $\int_{\mathbb{S}^{N-1}} \bar{m} dx = 0$. Set $m(\xi) := \bar{m}(\xi/|\xi|) \in C^\infty(\mathbb{R}^N \setminus \{0\})$. Then there exists $\Omega \in C^\infty(\mathbb{S}^{N-1})$ with $\int_{\mathbb{S}^{N-1}} \Omega dx = 0$ such that the Fourier transform of $m(\xi)$ is given (as distributions) by $\frac{\Omega(x/|x|)}{|x|^N}$. More precisely, let*

$$T_\Omega u(x) := \lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} \frac{\Omega(y/|y|)}{|y|^N} u(x-y)dy, \ \ \forall u \in C_0^\infty(\mathbb{R}^N),$$

then $\hat{T_\Omega} u(\xi) = m(\xi)\hat{u}(\xi), \forall \xi \in \mathbb{R}^N$.

With these preparations, we now return to the proof of the interior elliptic estimates.

Lemma 1.59. *Suppose L is of constant coefficients and homogeneous, i.e., $L = \sum_{|\alpha|=k} A_\alpha \partial^\alpha$ where A_α are constant $n \times n$ matrices. Let $\lambda > 0$ be the minimum of the norms of the matrices $\sigma(L)(\xi) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha$ over the unit sphere $|\xi| = 1$, and let $A := \max_\alpha \{|a_\alpha^{ij}|\}$ where a_α^{ij} are the entries of A_α, i.e., $A_\alpha = (a_\alpha^{ij})$. Let $u \in C_0^\infty(E)$. Then*

(1) for any $1 < p < \infty$, there exists $C(\lambda, A, p) > 0$ such that

$$\sum_{|\beta|=k} ||\partial^\beta u||_p \leq C(\lambda, A, p)||Lu||_p,$$

(2) for any $0 < \alpha < 1$, there exists $C(\lambda, A, \alpha, R) > 0$ such that

$$\sum_{|\beta|=k} ||\partial^\beta u||_{0,\alpha} \leq C(\lambda, A, \alpha, R)||Lu||_{0,\alpha},$$

if $u(x) = 0$ for $|x| \geq R$.

Proof. Let $v := Lu \in C_0^\infty(F)$. Write $u = (u_1, \cdots, u_n)^T$, $v = (v_1, \cdots, v_n)^T$, and let

$$\hat{u}(\xi) := (\hat{u}_1(\xi), \cdots, \hat{u}_n(\xi))^T, \hat{v}(\xi) := (\hat{v}_1(\xi), \cdots, \hat{v}_n(\xi))^T.$$

Then for any multi-index β with $|\beta| = k$, we have $L(\partial^\beta u) = \partial^\beta v$, and taking Fourier transforms of both sides, we obtain

$$(-2\pi i)^k (\sum_{|\alpha|=k} A_\alpha \xi^\alpha) \hat{\partial^\beta u}(\xi) = \xi^\beta \hat{v}(\xi), \quad \forall \xi \in \mathbb{R}^N.$$

For $\xi \neq 0$, we solve for $\hat{\partial^\beta u}(\xi)$

$$\hat{\partial^\beta u}(\xi) = (-2\pi i)^{-k} (\sum_{|\alpha|=k} A_\alpha \xi^\alpha)^{-1} \xi^\beta \hat{v}(\xi),$$

and therefore

$$\hat{\partial^\beta u_i}(\xi) = \sum_{j=1}^n m_{ij}(\xi) \hat{v}_j(\xi), \quad \forall i = 1, 2, \cdots, n,$$

where $m_{ij}(\xi)$ are homogeneous of degree 0 so that $m_{ij}(\xi) = \bar{m}_{ij}(\xi/|\xi|)$ for some $\bar{m}_{ij} \in C^\infty(\mathbb{S}^{N-1})$. Set $c_{ij} := \int_{\mathbb{S}^{N-1}} \bar{m}_{ij} dx$. Then applying Proposition 1.58 to $\bar{m}_{ij} - c_{ij}$, we obtain $\Omega_{ij} \in C^\infty(\mathbb{S}^{N-1})$ with $\int_{\mathbb{S}^{N-1}} \Omega_{ij} dx = 0$ and the associated operators T_{ij}.

By the Calderon-Zygmund inequality, for any $1 < p < \infty$,

$$||\partial^\beta u_i||_p \leq \sum_{j=1}^n (||T_{ij} v_j||_p + |c_{ij}|||v_j||_p) \leq C \sum_{j=1}^n ||v_j||_p, \forall i = 1, 2, \cdots, n,$$

which implies

$$\sum_{|\beta|=k} ||\partial^\beta u||_p \leq C||v||_p = C||Lu||_p,$$

where $C = C(\{\Omega_{ij}, |c_{ij}||i, j = 1, \cdots, n\}, p)$.

Similarly, using Hölder-Korn-Lichtenstein-Girand, one obtains for $0 < \alpha < 1$,

$$\sum_{|\beta|=k} ||\partial^\beta u||_{0,\alpha} \leq C||Lu||_{0,\alpha}$$

if $u(x) = 0$ for $|x| \geq R$, where $C = C(\{\Omega_{ij}, |c_{ij}||i, j = 1, \cdots, n\}, \alpha, R) > 0$.

Finally, to determine how the constant C may depend on L, we inspect the dependence of C on $\{\Omega_{ij}\}$ and that of Ω_{ij} on $\bar{m}_{ij} - c_{ij}$. We conclude that $C = C(\lambda, A, p)$ in the Sobolev case and $C = C(\lambda, A, \alpha, R)$ in the Hölder case. $\square$

With this lemma, we shall prove a version of Theorem 1.56 with an additional assumption that supp $u \subset D'$. Note that in this case, one has

$$||Lu||_{m,p,D'} = ||Lu||_{m,p,D}, ||u||_{p,D'} = ||u||_{p,D}, ||Lu||_{m,\alpha,D'} = ||Lu||_{m,\alpha,D}, ||u||_{0,\alpha,D'} = ||u||_{0,\alpha,D}.$$

We shall first deal with the case where $m = 0$. Let $L = \sum_{|\alpha|\leq k} A_\alpha(x)\partial^\alpha$, $x \in D$. We denote by $\lambda > 0$ the minimum of the norms of matrices $\sigma(L)(x)(\xi) = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha$ over $x \in \overline{D}'$ and the unit sphere $|\xi| = 1$, and let

$$A := \max_{x\in\overline{D}',\, |\alpha|=k} \{|a^{ij}_\alpha(x)|, |\nabla a^{ij}_\alpha(x)|\}, \quad B := \max_{x\in\overline{D}',\, |\alpha|<k} \{|a^{ij}_\alpha(x)|\},$$

where $A_\alpha(x) = (a^{ij}_\alpha(x))$.

Proposition 1.60. *Suppose $u \in C^\infty(E)$ such that supp $u \subset D'$. Then*

$$||u||_{k,p,D'} \leq C(||Lu||_{p,D'} + ||u||_{p,D'}),$$

where $C = C(\lambda, A, B, p, D') > 0$, $1 < p < \infty$.

Proof. Cover $\overline{D}'$ by finitely many balls $B_\nu := B_{x_\nu}(r)$ of radius $r > 0$ centered at $x_\nu \in D$, $\nu = 1, 2, \cdots, M$, with r to be specified later. For each ν, pick a bump function $\rho_\nu \geq 0$ with supp $\rho_\nu \subset B_\nu$ such that $\sum_{\nu=1}^M \rho_\nu = 1$ on $\overline{D}'$. Finally, set $u_\nu := \rho_\nu u$ and $L_\nu := \sum_{|\alpha|=k} A_\alpha(x_\nu)\partial^\alpha$. Note that supp $u_\nu \subset B_\nu$.

By Lemma 1.59(1), for each ν, $\sum_{|\beta|=k} ||\partial^\beta u_\nu||_{p,B_\nu} \leq C(\lambda, A, p)||L_\nu u_\nu||_{p,B_\nu}$. We write

$$\begin{aligned} L_\nu u_\nu &= Lu_\nu + (L_\nu - L)u_\nu \\ &= \rho_\nu Lu + [L, \rho_\nu]u + \sum_{|\alpha|=k} (A_\alpha(x_\nu) - A_\alpha(x))\partial^\alpha u_\nu + \sum_{|\alpha|<k} A_\alpha(x)\partial^\alpha u_\nu \end{aligned}$$

and observe the following estimates

$$||\rho_\nu Lu||_{p,B_\nu} \leq ||Lu||_{p,D'}, \quad ||[L, \rho_\nu]u||_{p,B_\nu} \leq C(r, A, B)||u||_{k-1,p,D'},$$

$$||\sum_{|\alpha|=k} (A_\alpha(x_\nu) - A_\alpha(x))\partial^\alpha u_\nu||_{p,B_\nu} \leq Ar \sum_{|\alpha|=k} ||\partial^\alpha u_\nu||_{p,B_\nu},$$

and

$$||\sum_{|\alpha|<k} A_\alpha(x)\partial^\alpha u_\nu||_{p,B_\nu} \leq C(B, r)||u||_{k-1,p,D'}.$$

Choose $r > 0$ so that $C(\lambda, A, p)Ar = 1/2$, one can "absorb" the term

$$||\sum_{|\alpha|=k} (A_\alpha(x_\nu) - A_\alpha(x))\partial^\alpha u_\nu||_{p,B_\nu}$$

in $||L_\nu u_\nu||_{p,B_\nu}$ and obtain

$$\sum_{|\beta|=k} ||\partial^\beta u_\nu||_{p,B_\nu} \leq C(||Lu||_{p,D'} + ||u||_{k-1,p,D'})$$

for a constant $C = C(\lambda, A, B, p, r) > 0$. Now over $\overline{D}'$, $u = \sum_{\nu=1}^{M} u_\nu$, and moreover, observe that $||u||_{k,p,D'} = \sum_{|\beta|=k} ||\partial^\beta u||_{p,D'} + ||u||_{k-1,p,D'}$, one has

$$\begin{aligned} ||u||_{k,p,D'} &\leq \sum_{\nu=1}^{M} \sum_{|\beta|=k} ||\partial^\beta u_\nu||_{p,B_\nu} + ||u||_{k-1,p,D'} \\ &\leq C(||Lu||_{p,D'} + ||u||_{k-1,p,D'}) \end{aligned}$$

for a constant $C = C(\lambda, A, B, p, r, M) > 0$. Finally, we use the interpolation inequalities

$$||u||_{k-1,p,D'} \leq \epsilon ||u||_{k,p,D'} + C(\epsilon)||u||_{p,D'}$$

to "absorb" the term $||u||_{k-1,p,D'}$ by choosing a sufficiently small $\epsilon > 0$.

□

Exercise 1.61. (1) Assume supp $u \subset D'$. Prove that for any integer $m \geq 0$,

$$||u||_{k+m,p,D'} \leq C(||Lu||_{m,p,D'} + ||u||_{p,D'}).$$

Also, explain the dependence of C on the coefficients of L. (Hint: For any multi-index β with $|\beta| = m$, $[L, \partial^\beta]$ is a p.d.o. of order $\leq k + m - 1$. Use interpolation inequalities to "absorb" it.)

(2) Do the case of Hölder spaces.

Remark 1.62. Without the assumption that supp $u \subset D'$, the argument in Proposition 1.60 still gives an estimate

$$||u||_{k,p,D'} \leq C(||Lu||_{p,D} + ||u||_{k-1,p,D}).$$

However, the interpolation inequality argument at the end breaks down.

In order to remove the assumption supp $u \subset D'$ and prove Theorem 1.56 in full generality, one has to apply the interpolation inequalities in a more subtle way. For simplicity, we shall only illustrate this for the case of $k = 2$. Essentially one needs to deal with the case where $D = B_R$, $D' = B_r$ are balls centered at 0 with radius $R > r > 0$ and R sufficiently small. For general D', D, a covering argument with small balls will do.

Proposition 1.63. *Assume $L = \sum_{|\alpha|\leq 2} A_\alpha(x)\partial^\alpha$ is of order 2 and elliptic. Then for sufficiently small $R > 0$, and any $0 < r < R$,*

$$||u||_{2,p,B_r} \leq C(||Lu||_{p,B_R} + ||u||_{p,B_R}), \ \ \forall u \in C^\infty(E)$$

where $C = C(\lambda, A, B, p, r, R) > 0$ and $1 < p < \infty$. Here λ is the minimum of the norms of the matrices $\sum_{|\alpha|=2} A_\alpha(0)\xi^\alpha$ over $|\xi| = 1$, $A := \max_{x\in\overline{B_R}, |\alpha|=2}\{|a_\alpha^{ij}(0)|, |\nabla a_\alpha^{ij}(x)|\}$, and $B := \max_{x\in\overline{B_R}, |\alpha|<2}\{|a_\alpha^{ij}(x)|\}$, where $A_\alpha(x) = (a_\alpha^{ij}(x))$.

Proof. For $0 < \sigma < 1$, pick a cut-off function η_σ such that $\eta_\sigma \equiv 1$ on $B_{\sigma R}$ and $\eta_\sigma \equiv 0$ outside $B_{\sigma' R}$ where $\sigma' = (1 + \sigma)/2 > \sigma$. Moreover, we can arrange so that $|\nabla \eta_\sigma| \leq 100((1 - \sigma)R)^{-1}$. Set $u_\sigma := \eta_\sigma u$.

As we argued in the proof of Proposition 1.60, when $R > 0$ is sufficiently small, one can absorb the term $||\sum_{|\alpha|=2}(A_\alpha(0) - A_\alpha(x))\partial^\alpha u_\sigma||_{p,B_R}$, and have

$$\begin{aligned}\sum_{|\alpha|=2} ||\partial^\alpha u||_{p,B_{\sigma R}} &\leq \sum_{|\alpha|=2} ||\partial^\alpha u_\sigma||_{p,B_R} \\ &\leq C(||Lu||_{p,B_R} + ||[L,\eta_\sigma]u||_{p,B_R} + ||\sum_{|\alpha|<2} A_\alpha(x)\partial^\alpha u_\sigma||_{p,B_R}).\end{aligned}$$

Notice that

$$||[L,\eta_\sigma]u||_{p,B_R} \leq C(((1-\sigma)R)^{-1}\sum_{|\alpha|=1} ||\partial^\alpha u||_{p,B_{\sigma' R}} + ((1-\sigma)R)^{-2}||u||_{p,B_{\sigma' R}}),$$

and

$$||\sum_{|\alpha|<2} A_\alpha(x)\partial^\alpha u_\sigma||_{p,B_R} \leq C(\sum_{|\alpha|=1} ||\partial^\alpha u||_{p,B_{\sigma' R}} + ((1-\sigma)R)^{-1}||u||_{p,B_{\sigma' R}}).$$

It follows, with the observation $(1-\sigma)/2 = 1-\sigma'$, that

$$((1-\sigma)R)^2 \sum_{|\alpha|=2} ||\partial^\alpha u||_{p,B_{\sigma R}} \leq C(||Lu||_{p,B_R} + (1-\sigma')R\sum_{|\alpha|=1} ||\partial^\alpha u||_{p,B_{\sigma' R}} + ||u||_{p,B_R}),$$

where $C = C(\lambda, A, B, p, R) > 0$.

We introduce

$$\Phi_l := \sup_{0<\sigma<1} ((1-\sigma)R)^l \sum_{|\alpha|=l} ||\partial^\alpha u||_{p,B_{\sigma R}} < \infty, \quad l = 0, 1, 2.$$

Then we have $\Phi_2 \leq C(||Lu||_{p,B_R} + \Phi_1 + ||u||_{p,B_R})$.

Next we will show that for any $\epsilon > 0$, there exists $C(\epsilon) > 0$, such that

$$\Phi_1 \leq \epsilon\Phi_2 + C(\epsilon)\Phi_0.$$

To this end, let $\gamma > 0$ be any number. Choose $\sigma = \sigma(\gamma)$ such that

$$\Phi_1 \leq (1-\sigma)R\sum_{|\alpha|=1} ||\partial^\alpha u||_{p,B_{\sigma R}} + \gamma.$$

Now by the interpolation inequality, $\forall \epsilon > 0$,

$$\sum_{|\alpha|=1} ||\partial^\alpha u||_{p,B_{\sigma R}} \leq \epsilon(1-\sigma)R\sum_{|\alpha|=2} ||\partial^\alpha u||_{p,B_{\sigma R}} + \frac{C}{\epsilon(1-\sigma)R}||u||_{p,B_{\sigma R}}.$$

This gives rise to

$$\begin{aligned}\Phi_1 &\leq \epsilon((1-\sigma)R)^2\sum_{|\alpha|=2} ||\partial^\alpha u||_{p,B_{\sigma R}} + \frac{C}{\epsilon}||u||_{p,B_{\sigma R}} + \gamma \\ &\leq \epsilon\Phi_2 + \frac{C}{\epsilon}\Phi_0 + \gamma.\end{aligned}$$

Letting $\gamma \to 0$, we have $\Phi_1 \leq \epsilon\Phi_2 + \frac{C}{\epsilon}\Phi_0$.

Now absorbing Φ_1, we obtain $\Phi_2 \leq C(||Lu||_{p,B_R} + ||u||_{p,B_R})$, and for any $0 < r < R$,

$$\sum_{|\alpha|=2} ||\partial^\alpha u||_{p,B_r} \leq \frac{1}{(R-r)^2}\Phi_2 \leq C(||Lu||_{p,B_R} + ||u||_{p,B_R}),$$

where $C = C(\lambda, A, B, p, R, r) > 0$. Finally, using interpolation inequality again,

$$\begin{aligned} ||u||_{2,p,B_r} &\leq C(\sum_{|\alpha|=2} ||\partial^\alpha u||_{p,B_r} + ||u||_{p,B_r}) \\ &\leq C(||Lu||_{p,B_R} + ||u||_{p,B_R}). \end{aligned}$$

□

Exercise 1.64. Complete the proof of Theorem 1.56.

Definition 1.65. (1) We say u is a *classical* solution of $Lu = v$ if $u \in C^k(E)$, $v \in C^0(F)$ and $Lu = v$ pointwise in D.

(2) We say u is a L^p *strong* solution of $Lu = v$ if $u \in L^{k,p}_{loc}(E)$, $v \in L^p_{loc}(F)$ where $1 \leq p < \infty$, and $Lu = v$ almost everywhere in D.

(3) We say u is a L^p *weak* solution of $Lu = v$ if $u \in L^p_{loc}(E)$, $v \in L^p_{loc}(F)$ for some $1 \leq p < \infty$, and $Lu = v$ weakly in D, i.e., $\forall \phi \in C_0^\infty(F)$,

$$\int_D \langle u, L^*\phi\rangle_E dx = \int_D \langle v, \phi\rangle_F dx.$$

Exercise 1.66. Show that

$$\text{Classical solutions} \quad \Rightarrow \quad \text{Strong solutions} \quad \Rightarrow \quad \text{Weak solutions}$$

The issue of regularity concerns whether the above arrows can be reversed, i.e.,

$$\text{Classical solutions} \quad \overset{?}{\Leftarrow} \quad \text{Strong solutions} \quad \overset{?}{\Leftarrow} \quad \text{Weak solutions}$$

Next we shall illustrate with a few examples how to prove or improve regularity of solutions of elliptic equations using mollifiers $\rho_\delta *$ and apriori estimates. A key issue of this approach is the behavior of the commutator $[L, \rho_\delta *]$. We shall first look at the case when L has constant coefficients, in which case $[L, \rho_\delta *] = 0$.

Proposition 1.67. *Suppose $L : C^\infty(E) \to C^\infty(F)$ is an elliptic p.d.o. of order $k > 0$ over D, which has constant coefficients. For $1 < p < \infty$, let $u \in L^p_{loc}(E)$, $v \in L^p_{loc}(F)$, where $Lu = v$ weakly in D. Then for any integer $m \geq 0$, if $v \in L^{m,p}_{loc}(F)$, then $u \in L^{m+k,p}_{loc}(E)$.*

Proof. Consider the case $m = 0$ first. Let $x_0 \in D$ be any point, and let $R > 0$ be small enough such that the ball B_{4R} of radius $4R$ centered at x_0 is contained in D. We pick a cut-off function η, $0 \leq \eta \leq 1$, such that $\eta \equiv 1$ on B_{2R}, $\eta \equiv 0$ outside B_{3R}. Let $\tilde{u} = \eta u$, $\tilde{v} = \eta v$. Then $\tilde{u} \in L^p(E|_{B_{4R}})$, $\tilde{v} \in L^p(F|_{B_{4R}})$. Moreover, For any $\delta < R$, the mollifiers $\rho_\delta * \tilde{u}$, $\rho_\delta * \tilde{v}$ converges to $\tilde{u}$, $\tilde{v}$ strongly in $L^p(E|_{B_{4R}})$, $L^p(F|_{B_{4R}})$ respectively. In particular, there is a constant $C_0 > 0$ independent of δ such that both $||\rho_\delta * \tilde{u}||_{p,B_{4R}} \leq C_0$, $||\rho_\delta * \tilde{v}||_{p,B_{4R}} \leq C_0$, $\forall \delta$.

Now by the interior elliptic estimates, for any $0 < r < R$,

$$||\rho_\delta * \tilde{u}||_{k,p,B_r} \leq C(||L(\rho_\delta * \tilde{u})||_{p,B_R} + ||\rho_\delta * \tilde{u}||_{p,B_R})$$

where $C > 0$ is also independent of δ. Note that when $\delta < R$, $\rho_\delta * \tilde{u} = \rho_\delta * u$, $\rho_\delta * \tilde{v} = \rho_\delta * v$ on B_R, so that

$$L(\rho_\delta * \tilde{u}) = L(\rho_\delta * u) = \rho_\delta * (Lu) = \rho_\delta * v = \rho_\delta * \tilde{v} \text{ on } B_R.$$

Consequently, we obtain the bound $||\rho_\delta * \tilde{u}||_{k,p,B_r} \leq 2CC_0$, $\forall \delta$.

At this point we need to recall a result from functional analysis, i.e., if $(f_n) \subset B$ is a sequence in a *reflexive* Banach space B such that $||f_n|| \leq C$ for some constant $C > 0$ for all n, then there is a $f \in B$ and a subsequence (f_{n_k}) such that $f_{n_k} \to f$ weakly in B, which means that for any functional $w \in B^*$ in the dual space of B, $\lim_{k\to\infty} w(f_{n_k}) = w(f)$, cf. [15].

Notice that $L^{k,p}(E|_{B_r})$ is reflexive. Hence there exists a sequence $\delta_n \to 0$, and a $\bar{u} \in L^{k,p}(E|_{B_r})$, such that $\rho_{\delta_n} * u \to \bar{u}$ weakly in $L^{k,p}(E|_{B_r})$. In particular, for any $\phi \in C_0^\infty(E|_{B_r})$,

$$\lim_{n\to\infty} \int_{B_r} \langle \rho_{\delta_n} * u, \phi\rangle_E dx = \int_{B_r} \langle \bar{u}, \phi\rangle_E dx.$$

On the other hand, since $\rho_{\delta_n} * u \to u$ strongly in $L^p(E|_{B_r})$, we have

$$\lim_{n\to\infty} \int_{B_r} \langle \rho_{\delta_n} * u, \phi\rangle_E dx = \int_{B_r} \langle u, \phi\rangle_E dx,$$

which implies $\int_{B_r} \langle \bar{u}, \phi\rangle_E dx = \int_{B_r} \langle u, \phi\rangle_E dx$, $\forall \phi \in C_0^\infty(E|_{B_r})$. This implies that $\bar{u} = u$ almost everywhere in B_r. Hence $u \in L^{k,p}(E|_{B_r})$ for sufficiently small $r > 0$. Since $x_0 \in D$ is arbitrary, we conclude that $u \in L^{k,p}_{loc}(E)$.

For $m > 0$, notice $[L, \partial^\alpha] = 0$ and use induction on m.

□

The above argument clearly works for L with more general coefficients, as long as one can get a contral over the commutator $[L, \rho_\delta *]$. The next lemma looks at the case where L is a first order p.d.o.

Lemma 1.68. *Suppose L is a first order p.d.o. (not necessarily elliptic). Let $u, v \in L^p$ and $Lu = v$ weakly. Then there exists a constant $C > 0$ independent of δ such that $||L(\rho_\delta * u) - \rho_\delta * v||_p \leq C$, $\forall \delta$.*

Proof. For simplicity we assume E, F are of rank 1. It suffices to consider the case where $L = a(x)\partial$, $a(x)$ a smooth function. With this understood,

$$L(\rho_\delta * u)(x) = a(x)\partial_x(\int \rho_\delta(x-y)u(y)dy) = \int \partial_x\rho_\delta(x-y)a(x)u(y)dy = -\int \partial_y\rho_\delta(x-y)a(x)u(y)dy.$$

On the other hand, since $Lu = v$ weakly, and notice that $\rho_\delta \in C_0^\infty$, one has

$$\rho_\delta * v(x) = \int \rho_\delta(x-y)v(y)dy = -\int \partial_y(a(y)\rho_\delta(x-y))u(y)dy.$$

Hence $L(\rho_\delta * u)(x) - \rho_\delta * v(x) = \int \partial_y((a(y) - a(x))\rho_\delta(x-y))u(y)dy$. We write

$$\partial_y((a(y) - a(x))\rho_\delta(x-y)) = \partial_y(a(y))\rho_\delta(x-y) + (a(y) - a(x))\partial_y\rho_\delta(x-y),$$

and observe $|(a(y)-a(x))\partial_y\rho_\delta(x-y)| \leq \sup|\nabla a|\cdot\delta^{-N}|\partial\rho|((x-y)/\delta)$. Hence

$$\begin{aligned}||L(\rho_\delta * u)-\rho_\delta * v||_p &\leq ||\rho_\delta * |u\cdot\partial a|||_p + \sup|\nabla a|\cdot|||\partial\rho|_\delta * |u|||_p \\ &\leq ||\rho_\delta||_1\cdot||u\cdot\partial a||_p + \sup|\nabla a|\cdot|||\partial\rho|_\delta||_1\cdot||u||_p \text{ (by Young's ineq.)} \\ &\leq \sup|\nabla a|(1+|||\partial\rho|||_1)\cdot||u||_p\end{aligned}$$

which is independent of δ. Note that in the last step we used $||\rho_\delta||_1 = ||\rho||_1 = 1$ and $|||\partial\rho|_\delta||_1 = |||\partial\rho|||_1$. ☐

Proposition 1.69. *Suppose $L : C^\infty(E) \to C^\infty(F)$ is a first order elliptic p.d.o. over D. For $1 < p < \infty$, let $u \in L^p_{loc}(E)$, $v \in L^p_{loc}(F)$, where $Lu = v$ weakly in D. Then for any integer $m \geq 0$, if $v \in L^{m,p}_{loc}(F)$, then $u \in L^{m+k,p}_{loc}(E)$.*

Exercise 1.70. (1) Prove Proposition 1.69.

(2) Prove a generalization of Proposition 1.69 which says: if L is of order $k > 0$ and $u \in L^{(k-1),p}_{loc}(E), v \in L^p_{loc}(F)$, then $Lu = v$ weakly implies that $u \in L^{k,p}_{loc}(E)$.

Here is another example.

Exercise 1.71. Let $L = \sum_{|\alpha|\leq k} A_\alpha(x)\partial^\alpha : C^\infty(E) \to C^\infty(F)$ be an elliptic p.d.o. of order $k > 0$ over D. Suppose $u \in C^k(E)$. If $Lu \in C^{0,\alpha}(F)$ for some $0 < \alpha < 1$, then $u \in C^{k,\alpha}(E)$.

Hints: (1) There exists $C_0 > 0$ such that $||\rho_\delta * u||_{0,\alpha} \leq C_0||u||_{0,\alpha}$, $||\rho_\delta * Lu||_{0,\alpha} \leq C_0||Lu||_{0,\alpha}$, $\forall\delta$.

(2) $||L(\rho_\delta * u) - \rho_\delta * Lu||_{0,\alpha} \leq C_1 \max_\alpha ||A_\alpha||_{0,\alpha}\cdot||u||_{C^k}$ for some $C_1 > 0$, $\forall\delta$.

Note that (1), (2) plus interior elliptic estimates imply that $||\rho_\delta * u||_{k,\alpha} \leq C_2(||Lu||_{0,\alpha} + ||u||_{C^k})$ for some $C_2 > 0$, $\forall\delta$.

(3) Use the fact that $\rho_\delta * u$ converges to u uniformly in $C^k(E)$ to show that $||u||_{k,\alpha} \leq C_2(||Lu||_{0,\alpha} + ||u||_{C^k})$. In particular, $u \in C^{k,\alpha}(E)$.

There are different approaches to a proof of the following theorem, one of which is through pseudo-differential operators, cf. e.g. [7].

Theorem 1.72. *(Regularity of L^p Weak Solutions) Suppose $L : C^\infty(E) \to C^\infty(F)$ is an elliptic p.d.o. of order $k > 0$ over D. For $1 < p < \infty$, let $u \in L^p_{loc}(E)$, $v \in L^p_{loc}(F)$, where $Lu = v$ weakly in D. Then for any integer $m \geq 0$, if $v \in L^{m,p}_{loc}(F)$, then $u \in L^{m+k,p}_{loc}(E)$. Moreover, if $v \in C^{m,\alpha}_{loc}(F)$ for some $0 < \alpha < 1$, then $u \in C^{m+k,\alpha}_{loc}(E)$.*

Corollary 1.73. *If u is a L^p weak solution of $Lu = v$ for some $1 < p < \infty$ where v is smooth, then u must be smooth.*

1.4. **Elliptic operators on compact manifolds.** Let $L : C^\infty(E) \to C^\infty(F)$ be an elliptic p.d.o. of order $k > 0$ over a *compact*, oriented manifold M. We fix a $1 < p < \infty$. For any integer $m \geq 0$, L determines uniquely a bounded linear operator

$$L_m : L^{m+k,p}(E) \to L^{m,p}(F),$$

because (1) there exists $C > 0$ such that $\forall u \in C^\infty(E)$, $||Lu||_{m,p} \leq C||u||_{m+k,p}$, and (2) $C^\infty(E) \subset L^{m+k,p}(E)$ is a dense subspace. Note that $\forall u \in L^{m+k,p}(E)$, $Lu = v \in L^{m,p}(F)$ as a L^p strong solution, and moreover, $L_m u = Lu$.

The first goal of this section is to establish the Fredholm properties of L_m. The proof is based on what we discussed in sections 1.2 and 1.3. First of all, we observe

Proposition 1.74. *There exists $C > 0$ such that*

$$||u||_{m+k,p} \leq C(||Lu||_{m,p} + ||u||_p), \quad \forall u \in L^{m+k,p}(E).$$

Proof. Since M is compact, we can cover M by finitely many coordinate balls $B_{x_\nu}(R)$ of radius R centered at $x_\nu \in M$ for some $R > 0$, $\nu = 1, 2, \cdots, S$. Furthermore, we can assume $B_{x_\nu}(2R)$ of radius $2R$ is also a coordinate ball. By the interior elliptic estimates, there exists $C' > 0$ such that for all ν,

$$||u||_{m+k,p,B_{x_\nu}(R)} \leq C'(||Lu||_{m,p,B_{x_\nu}(2R)} + ||u||_{p,B_{x_\nu}(2R)}), \quad \forall u \in C^\infty(E).$$

This gives

$$||u||_{m+k,p} \leq \sum_{\nu=1}^{S} ||u||_{m+k,p,B_{x_\nu}(R)} \leq SC'(||Lu||_{m,p} + ||u||_p)$$

for any $u \in C^\infty(E)$. Taking $C = SC'$, the proposition follows by the density of $C^\infty(E)$ in $L^{m+k,p}(E)$. □

Recall that the kernel of L_m is $\ker L_m := \{u \in L^{m+k,p}(E) | Lu = 0 \in L^{m,p}(F)\}$. The image of L_m is Im $L_m := \{v \in L^{m,p}(F) | v = Lu \text{ for some } u \in L^{m+k,p}(E)\} \subset L^{m,p}(F)$. The cokernel of L_m is coker $L_m := L^{m,p}(F)/\text{Im } L_m$. Note that by the elliptic regularity, $\ker L_m = \ker L$, where $L : C^\infty(E) \to C^\infty(F)$. The main properties (i.e. Fredholm) of L_m are listed in the following theorem.

Theorem 1.75. *(1)* $\ker L_m = \ker L$ *and* $\dim \ker L_m < \infty$.

(2) The image Im $L_m \subset L^{m,p}(F)$ is a closed subspace.

(3) The dual space of coker L_m is given by $\ker L^*$ *through the L^2 inner product, where $L^* : C^\infty(F) \to C^\infty(E)$ is the formal adjoint of L. In particular,* $\dim$ *coker* $L_m < \infty$.

(4) There is a decomposition $L^{m,p}(F) = \ker L^* \oplus$ Im L_m, which is orthogonal with respect to the L^2 inner product.

Definition 1.76. (Fredholm Operators) A linear operator $F : V \to W$ between Banach spaces is called *Fredholm* if (1) $\dim \ker F < \infty$, and Im $F \subset W$ is closed, (2) $\dim \text{coker } F < \infty$, where coker $F := W/\text{Im } F$. The index of F is defined to be

$$\text{Index } F := \dim \ker F - \dim \text{coker } F.$$

Remark 1.77. (1) By Theorem 1.75, for any $1 < p < \infty$ and $m \geq 0$, $L_m : L^{m+k,p}(E) \to L^{m,p}(F)$ is a Fredholm operator. Moreover, Index $L_m = \dim \ker L - \dim \ker L^*$, which is independent of p and m, and is simply defined to be the index of L, i.e.,

$$\text{Index } L := \text{Index } L_m = \dim \ker L - \dim \ker L^*.$$

In fact, Index L may be computed in terms of topological invariants by the Atiyah-Singer Index Theorem, cf. [1].

(2) By Theorem 1.75(3), a $v \in L^{m,p}(F)$ lies in Im L_m if and only if

$$\int_M \langle v, \phi \rangle_F dVol_g = 0, \text{ for any } \phi \in \ker L^* \subset C^\infty(F).$$

Now we give a proof for Theorem 1.75. For part (1), note that as a closed subspace, $\ker L_m \subset L^{m+k,p}(E)$ is a Banach space under the norm $||\cdot||_{m+k,p}$. Let $B \subset \ker L_m$ be the unit ball, i.e., $B = \{u \in \ker L_m | ||u||_{m+k,p} \leq 1\}$. We will show that B is a compact subset, hence by the classical lemma of F. Riesz: the unit ball of an infinite dimensional Banach space is not compact, we conclude that $\dim \ker L_m < \infty$. To see that B is compact, note that by the Rellich-Kondrachove compactness theorem, B is precompact in the topology of L^p norm. But by Proposition 1.74, $||u||_{m+k,p} \leq C(||Lu||_{m,p} + ||u||_p) = C||u||_p$ for any $u \in \ker L_m$, so that the precompactness in L^p-topology implies precompactness in the original topology defined by $||\cdot||_{m+k,p}$. Hence B is compact.

For part (2), we consider the L^2-orthogonal complement of $\ker L_m$ in $L^{m+k,p}(E)$, i.e.,

$$(\ker L_m)^\perp := \{u \in L^{m+k,p}(E) | \int_M \langle u, \phi\rangle_E dVol = 0, \forall \phi \in \ker L_m\}.$$

Lemma 1.78. *(Poincaré inequality) There exists a constant $C > 0$ such that*

$$||u||_p \leq C||Lu||_{m,p}, \ \ \forall u \in (\ker L_m)^\perp.$$

Proof. Suppose there exists no such constants. Then there is a sequence $(u_n) \subset (\ker L_m)^\perp$ such that $||u_n||_p = 1$, $\forall n$, and $\lim_{n\to\infty} Lu_n = 0$ in $L^{m,p}(F)$. By the elliptic estimate (Prop. 1.74), $||u_n||_{m+k,p} \leq C_0(||Lu_n||_{m,p} + ||u_n||_p) \leq C_1$, $\forall n$. Hence by Rellich-Kondrachove, a subsequence of (u_n), still denoted by (u_n) for simplicity, converges to a $\bar{u}$ in the L^p-topology. Note that $||\bar{u}||_p = \lim_{n\to\infty} ||u_n||_p = 1$. In particular, $\bar{u} \neq 0$. We claim $\bar{u} \in \ker L_m$. To see this, note that for any $\phi \in C^\infty(F)$,

$$\int_M \langle L\bar{u}, \phi\rangle_F dVol = \int_M \langle \bar{u}, L^*\phi\rangle_E dVol = \lim_{n\to\infty} \int_M \langle u_n, L^*\phi\rangle_E dVol = \lim_{n\to\infty} \int_M \langle Lu_n, \phi\rangle_F dVol = 0,$$

which shows that $L\bar{u} = 0$ weakly. By elliptic regularity, $\bar{u} \in \ker L_m$.

A contradiction is reached as follows. Since $u_n \in (\ker L_m)^\perp$, we have

$$||\bar{u}||_2^2 = -\int_M \langle u_n - \bar{u}, \bar{u}\rangle_E dVol \leq ||u_n - \bar{u}||_p ||\bar{u}||_q, \text{ where } q = p/(p-1),$$

which implies $\bar{u} = 0$. ∎

With the above lemma, it follows immediately that Im L_m is closed. Suppose Lu_n converges to v in $L^{m,p}(F)$ (without loss of generality we assume $u_n \in (\ker L_m)^\perp$). Then (Lu_n) is a Cauchy sequence in $L^{m,p}(F)$, which implies that (u_n) is Cauchy in $L^{m+k,p}(E)$ by Lemma 1.78 and Prop. 1.74. Hence $u_n \to u$ in $L^{m+k,p}(E)$ and $Lu = v$.

For part (3) & (4), we first observe that the dual space of coker $L_m = L^{m,p}(F)/\text{Im } L_m$ can be identified with the subspace of $(L^{m,p}(F))^*$ which vanishes on Im L_m. With this understood, there is an embedding $\iota : \ker L^* \to (L^{m,p}(F)/\text{Im } L_m)^*$ via the L^2 inner product, i.e., $\iota(\phi)(v) = \int_M \langle v, \phi\rangle_F dVol$, $\forall \phi \in \ker L^*, v \in L^{m,p}(F)$. To see ι is onto, we first assume $m = 0$. Then since $(L^p(F))^* = L^q(F)$ via the L^2-product, where $q = p/(p-1) > 1$, we see in the case of $m = 0$, the dual space can be identified with the space of $v \in L^q(F)$ such that $L^*v = 0$ weakly. By elliptic regularity, $v \in C^\infty(F)$ and $v \in \ker L^*$.

Before proceeding to the case of $m > 0$ in general, we note that $\ker L^*$ is a subspace of $L^{m,p}(F)$ which is L^2-orthogonal to Im L_m. In particular, $\ker L^* \cap \mathrm{Im}\ L_m = \{0\}$. We have shown that (coker $L_m)^* = \ker L^*$ for the case $m = 0$ above. This gives in particular $L^{m,p}(F) = \ker L^* \oplus \mathrm{Im}\ L_m$ for $m = 0$. We claim it holds for $m > 0$ as well. To see this, let $v \in L^{m,p}(F)$. Then as an element of $L^p(F)$, $v = \phi + Lu$ where $\phi \in \ker L^*$ and $u \in L^{k,p}(E)$. Note that $Lu = v - \phi \in L^{m,p}(F)$, so that by the elliptic regularity, $u \in L^{m+k,p}(E)$. This shows that $L^{m,p}(F) = \ker L^* \oplus \mathrm{Im}\ L_m$, which is part (4). Finally, with the above decomposition, the embedding $\iota : \ker L^* \to (L^{m,p}(F)/\mathrm{Im}\ L_m)^*$ must be onto for any $m \geq 0$ by a dimension counting, which is part (3).

Exercise 1.79. Show that for any $0 < \alpha < 1$, $m \geq 0$, an elliptic p.d.o. L of order $k > 0$ defines a Fredholm operator between the Hölder spaces $C^{m+k,\alpha}(E)$ and $C^{m,\alpha}(F)$, whose index is given by $\dim\ker L - \dim\ker L^*$.

As a corollary of Theorem 1.75, we obtain the following

Corollary 1.80. *(Abstract Hodge Decomposition) Let $L : C^\infty(E) \to C^\infty(E)$ be a formally self-adjoint elliptic p.d.o of order $k > 0$. There is an orthogonal decomposition*

$$L^2(E) = \ker L \oplus Im\ L,$$

where $L : L^{k,2}(E) \to L^2(E)$.

Example 1.81. (Hodge Theory) (1) Let (M, g) be a compact, oriented Riemannian manifold of dimension N. The Hodge-Laplacian $\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M)$, $0 \leq k \leq N$, is a second order, formally self-adjoint, elliptic operator. By the Hodge Decomposition, every k-form α of L^2-class can be uniquely written as

$$\alpha = \alpha_0 + \Delta\beta = \alpha_0 + d\beta_1 + d^*\beta_2 \text{ (an orthogonal decomposition).}$$

Here $\alpha_0 \in \ker\Delta \subset \Omega^k(M)$ (called a *harmonic* k-form, which is equivalently characterized by $d\alpha_0 = d^*\alpha_0 = 0$), β is a k-form of $L^{2,2}$-class, and $\beta_1 = d^*\beta$, $\beta_2 = d\beta$. By elliptic regularity, both β_1, β_2 are smooth if α is smooth. As a consequence, one obtains the following

Hodge Theorem: on a compact, oriented Riemannian manifold (M, g), every de Rham cohomology class of degree k is uniquely represented by a harmonic k-form.

(2) Let M be a compact complex manifold of dimension n, and let g be a Hermitian metric on M. For any $0 \leq p \leq n$, consider the Dolbeault complex (which is elliptic)

$$0 \to \Omega^{p,0}(M) \xrightarrow{\bar\partial} \Omega^{p,1}(M) \xrightarrow{\bar\partial} \Omega^{p,2}(M) \to \cdots \to \Omega^{p,n}(M) \to 0.$$

The Dolbeault cohomology group $\mathcal{H}^{p,q}(M)$ is defined (similarly as the de Rham cohomology group) to be

$$\mathcal{H}^{p,q}(M) := \frac{\ker\{\Omega^{p,q}(M) \xrightarrow{\bar\partial} \Omega^{p,q+1}(M)\}}{\mathrm{Im}\ \{\Omega^{p,q-1}(M) \xrightarrow{\bar\partial} \Omega^{p,q}(M)\}}$$

Let $\bar\partial^* : \Omega^{p,*+1}(M) \to \Omega^{p,*}(M)$ be the formal adjoint of $\bar\partial$ (defined using the Hermitian metric g), and let $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ be the associated Laplacian. Then similar to the discussion in (1) above, the Hodge theory gives an identification of the Dolbeault cohomology group $\mathcal{H}^{p,q}(M)$ with $\ker\Delta_{\bar\partial}|_{\Omega^{p,q}(M)}$. Now suppose (M, g) is Kähler, which means that the associated 2-form $\omega(\cdot,\cdot) := -g(\cdot, J(\cdot))$ is closed (here J is the complex structure on M). Under this condition, one has the relation $\Delta_{\bar\partial} = \frac{1}{2}\Delta_d = \frac{1}{2}(dd^* + d^*d)$ between the corresponding Laplacians. As a consequence, the Hodge theory gives

$$H^k_{dR}(M) \otimes \mathbb{C} = \oplus_{p+q=k} \mathcal{H}^{p,q}(M), \;\; \forall 0 \leq k \leq 2n.$$

Moreover, one has $\mathcal{H}^{p,q}(M) = \overline{\mathcal{H}^{q,p}(M)}$ (the complex conjugate). Note that a particular corollary of the above identities is the following topological constraint of compact Kähler manifolds: *the odd Betti numbers of a compact Kähler manifold must be even.*

Example 1.82. (1) (The Euler Characteristic) Let (M, g) be a compact, oriented Riemannian manifold. The Hodge-de Rham operator $\delta = d + d^* : \Omega^*(M) \to \Omega^*(M)$ is formally self-adjoint, so its index is zero. To obtain something interesting, consider the decomposition $\Omega^*(M) = \Omega^{ev} \oplus \Omega^{odd}$ into forms of even and odd degrees. Then under this decomposition, $\delta = D \oplus D^*$, where $D := \delta|_{\Omega^{ev}}$ and D^* is the formal adjoint of D. Note that $D^* = \delta|_{\Omega^{odd}}$. One can verify easily that D is an elliptic p.d.o. The Atiyah-Singer index theorem computes the index of D, which gives

$$\text{Index } D = e(M)[M].$$

Here $e(M)$ is the Euler class of M, which via Chern-Weil theory can be expressed in terms of the curvatures of (M, g). On the other hand, by Hodge theory,

$$\text{Index } D = \dim \delta|_{\Omega^{ev}} - \dim \delta|_{\Omega^{odd}} = \sum_{k=ev} \dim H^k_{dR}(M) - \sum_{k=odd} \dim H^k_{dR}(M) = \chi(M),$$

which is the Euler characteristic $\chi(M)$. Hence one obtains $\chi(M) = e(M)[M]$. This formula predates the Atiyah-Singer, and is the higher-dimensional version of the classical Gauss-Bonnet Theorem: let Σ be a compact Riemann surface of genus g_Σ, then $2 - 2g_\Sigma = \frac{1}{2\pi}\int_\Sigma K dA$, where K is the Gaussian curvature and dA is the area form (of any given metric).

(2) (The Signature) Let (M, g) be a compact, oriented Riemannian manifold of dimension $N = 4l$. The cup product defines a symmetric bilinear form on the middle dimensional cohomology $H^{2l}(M) = H^{2l}_{dR}(M)$:

$$(\cdot,\cdot) : H^{2l}(M) \times H^{2l}(M) \to \mathbb{R}, \quad (\alpha, \beta) := \alpha \wedge \beta[M].$$

The signature of $(\cdot,\cdot)$ is called *the Signature* of M, and is denoted by Sign (M), which is a very important topological invariant of the manifold. Through Hodge theory, there is a way to express Sign (M) as the index of a certain elliptic p.d.o. on M. To this end, consider the involution $\tau : \Omega^*(M) \otimes \mathbb{C} \to \Omega^*(M) \otimes \mathbb{C}$ defined by $\tau(\alpha) = \sqrt{-1}^{p(p-1)+l} * \alpha$, $\forall \alpha \in \Omega^p(M)$. The involution τ gives rise to a decomposition of $\Omega^*(M) \otimes \mathbb{C} = \Omega^+ \oplus \Omega^-$ into the ± 1 eigenspaces of τ. Moreover, τ anti-commutes with δ, so that under the above decomposition, $\delta = D \oplus D^*$, where $D = \delta|_{\Omega^+}$ and its formal adjoint $D^* = \delta|_{\Omega^-}$. One can check that D is elliptic and Index $D =$ Sign (M).

The Atiyah-Singer index theorem computes the index of D and gives

$$\text{Sign } (M) = \mathcal{L}(M)[M],$$

where $\mathcal{L}(M)$ is a certain characteristic class of M, called the L-genus, which is expressed in terms of the Pontrjagin classes of M. The above formula was previously known and was due to F. Hirzebruch, called the Hirzebruch Signature Theorem. When $\dim M = 4$, it gives

$$\text{Sign }(M) = \frac{1}{3}p_1(M)[M], \text{ where } p_1 \text{ is the first Pontrjagin class.}$$

If furthermore, M is almost complex and let $c_1(M)$ be the first Chern class of the complex tangent bundle of M. Then the Chern-Weil theory gives an identity $c_1^2(M) = 2e(M) + p_1(M)$. By the above index formulas, one arrives at the following useful topological relation:

$$c_1^2(M)[M] = 2\chi(M) + 3 \text{ Sign }(M), \quad \text{where } M \text{ is an almost complex 4-manifold.}$$

(3) (Riemann-Roch) Let E be a holomorphic vector bundle over a compact Riemann surface Σ of genus g_Σ. The Cauchy-Riemann operator $\bar\partial : C^\infty(E) \to C^\infty(\Lambda^{0,1}\Sigma \otimes E)$ is a first order elliptic p.d.o. The Atiyah-Singer index theorem recovers, in this case, a formula due to Riemann-Roch:

$$\text{Index } \bar\partial = n(1 - g_\Sigma) + c_1(E)[\Sigma], \text{ where } n = \text{rank }_{\mathbb{C}}E.$$

The Riemann-Roch formula will be used in the calculation of the dimension of the moduli space of pseudo-holomorphic curves in Lecture 2.

The rest of this section will be devoted to the spectral theory of formally self-adjoint elliptic p.d.o.-s on compact manifolds.

Let $L : C^\infty(E) \to C^\infty(E)$ be an elliptic p.d.o. of order $k > 0$ over M, which is formally self-adjoint, i.e., $L = L^*$. As we have seen, L defines a Fredholm operator from $L^{k,2}(E)$ to $L^2(E)$, which is still denoted by L for simplicity. Note that $L^{k,2}(E) \subset L^2(E)$ is a dense subspace. In fact, we shall consider L as an operator from $L^2(E)$ to $L^2(E)$, but only defined on a dense subspace $D(L) = L^{k,2}(E)$. In this sense, L is not a bounded operator, but it is closed, meaning that its graph is closed. For simplicity, we shall denote the norm of $L^2(E)$ by $||\cdot||$ and the inner product by $\langle\cdot,\cdot\rangle$. With these notations, note that $L = L^*$ implies

$$\langle Lu, v\rangle = \langle u, Lv\rangle, \quad u, v \in D(L) = L^{k,2}(E).$$

First we recall some basic definitions. Let T be a closed linear operator defined on a subspace $D(T) \subset X$ of a Banach space X into X itself. The *resolvent set* of T consists of complex numbers $\lambda \in \mathbb{C}$ such that the operator $\lambda - T$ is invertible with an inverse which is a bounded linear operator on X. The complement of the resolvent set of T is called the *spectrum* of T, denoted by spec (T). A complex number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if $\ker(\lambda - T) \neq 0$. The space $\ker(\lambda - T)$ is called the *eigenspace* for λ and its dimension is called the *multiplicity* of λ, cf. [8].

Theorem 1.83. *Let $L : C^\infty(E) \to C^\infty(E)$ be a formally self-adjoint elliptic p.d.o. on a compact manifold. Then*

(1) The spectrum spec (L) consists of eigenvalues of L, each of which has finite multiplicity. Moreover, for each $\lambda \in$ spec (L), the eigenspace $\ker(\lambda - L)$ *consists of smooth sections of E.*

(2) The spectrum spec (L) is an unbounded subset of $\mathbb{R}$ with no accumulation points.

(3) There exists an orthogonal decomposition

$$L^2(E) = \oplus_{\lambda \in spec\ (L)} \ker(\lambda - L).$$

(4) Denote by P_λ the orthogonal projection onto $\ker(\lambda - L)$. Then

$$L^{k,2}(E) = \{u \in L^2(E) | \sum_{\lambda \in spec\ (L)} \lambda^2 ||P_\lambda u||^2 < \infty\}.$$

Note that in particular, (3), (4) imply that $u = \sum_\lambda P_\lambda u$, $\forall u \in L^2(E)$, and $Lu = \sum_\lambda \lambda P_\lambda u$, $\forall u \in D(L) = L^{k,2}(E)$.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of L. Then there exists a $0 \neq u \in D(L)$ such that $Lu = \lambda u$. Now

$$\lambda \langle u, u\rangle = \langle Lu, u\rangle = \langle u, Lu\rangle = \bar{\lambda}\langle u, u\rangle,$$

which implies $\lambda \in \mathbb{R}$ since $u \neq 0$. Furthermore, note that $\lambda - L$ is a formally self-adjoint elliptic p.d.o, so that by Corollary 1.80, $\lambda - L$ is not surjective. This shows that an eigenvalue of L lies in the spectrum spec (L). It follows from Theorem 1.75 that the space $\ker(\lambda - L)$ has finite dimension and consists of smooth sections of E.

To see that spec (L) consists entirely of eigenvalues, let $\lambda \in \mathbb{C}$ such that $\ker(\lambda - L) = 0$. Then $\ker(\lambda - L)^* = \ker(\bar{\lambda} - L) = 0$ also. This implies by Theorem 1.75 that $\lambda - L$ is invertible. Moreover, let T_λ be the inverse of $\lambda - L$. Then by Lemma 1.78, since $\ker(\lambda - L) = 0$, $T_\lambda : L^2(E) \to L^2(E)$ is a bounded operator. This shows that λ lies in the resolvent set of L and our claim follows.

For (2) and (3), we note that the operator T_λ defined above is in fact a compact operator, which exists, e.g., when λ is not a real number. To see T_λ is compact, note that by Prop. 1.74 and Lemma 1.78, there exists a $C(\lambda) > 0$ such that $||T_\lambda u||_{k,2} \leq C(\lambda)||u||$. This means that the image of a ball in $L^2(E)$ under T_λ has a bounded $L^{k,2}$-norm so that by Rellich-Kondrachov, it must be precompact in $L^2(E)$. Hence T_λ is compact. Furthermore, note that T_λ is normal, i.e., $T^*_\lambda T_\lambda = T_\lambda T^*_\lambda$.

Now we recall the following result (cf. Theorem 6.26 in p. 185 and Theorem 2.10 in p. 260 of [8]) from functional analysis: Let $T : V \to V$ be a normal, compact linear operator from a Hilbert space V into itself. Then the spectrum of T is a bounded, countable subset of $\mathbb{C}$ which has no accumulation point different from zero. Moreover, any $\mu \in \text{spec}\ (T) \setminus \{0\}$ is an eigenvalue of T with finite multiplicity, and there is an orthogonal decomposition $V = \oplus_{\mu \in \text{spec}\ (T) \setminus \{0\}} \ker(\mu - T) \oplus \ker T$.

With this understood, we pick a $\lambda_0 \in \mathbb{C} \setminus \text{spec}\ (L)$. Then (2) and (3) follow immediately, observing that $\ker T_{\lambda_0} = 0$,

$$\text{spec}\ (L) = \{\lambda_0 - \mu^{-1} | \mu \in \text{spec}\ (T_{\lambda_0}) \setminus \{0\}\}, \text{ and } \ker(\lambda - L) = \ker((\lambda_0 - \lambda)^{-1} - T_{\lambda_0}).$$

It remains to prove (4). First note that if $u \in L^{k,2}(E)$, then $Lu \in L^2(E)$, so that

$$\sum_{\lambda \in \text{spec}\ (L)} \lambda^2 ||P_\lambda u||^2 = ||\sum_\lambda \lambda P_\lambda u||^2 = ||Lu||^2 < \infty.$$

Conversely, if $u \in L^2(E)$ such that $\sum_{\lambda \in \text{spec}\ (L)} \lambda^2 ||P_\lambda u||^2 < \infty$, we consider the sequence of smooth sections $v_n := \sum_{|\lambda| \leq n} \lambda P_\lambda u$, which converges to $v := \sum_{|\lambda} \lambda P_\lambda u$ in $L^2(E)$. On the other hand, $v_n = Lu_n$ where $u_n := \sum_{|\lambda| \leq n} P_\lambda u$, which converges to u

in $L^2(E)$. By the elliptic estimate in Prop. 1.74, (u_n) is a Cauchy sequence in $L^{k,2}(E)$. It follows that $u = \lim_{n\to\infty} u_n \in L^{k,2}(E)$. $\square$

2. Non-linear Elliptic Equations

2.1. **Banach manifolds and Fredholm operators.** We begin by reviewing the basic notions of Banach manifolds (cf. [9]). Let E, F be Banach spaces, and let $L(E,F)$ denote the space of bounded linear operators from E to F, which is a Banach space under the operator norm: for $A \in L(E,F)$, $||A|| = \sup_{\{x\in E, ||x||=1\}} ||Ax||$.

Definition 2.1. Let $U \subset E$ be an open subset. A map $f : U \to F$ is *differentiable* at $x_0 \in U$ if there exists a $A \in L(E,F)$ such that

$$\lim_{h\to 0} \frac{||f(x_0+h) - f(x_0) - Ah||}{||h||} = 0, \ \ \forall h \in E.$$

The operator A is called the *derivative or differential* of f at x_0, which will be denoted by $Df(x_0)$ or Df_{x_0}.

Suppose f is differentiable at every point in U. Then the map $Df : U \to L(E,F)$, $x \mapsto Df(x)$, is defined, which is a map from U to a Banach space $L(E,F)$. We define the $2nd$ *derivative* of f to be the derivative of Df (if it exists), and denote it by $D^2 f \in L(E, L(E,F))$. One can continue with this process and define inductively the l-th *derivative* $D^l f$ of f for any $l \geq 1$. We say that f is of *class* C^l on U if $D^l f$ exists and is continuous on U. We say f is *smooth* (or of class C^∞) if it is of class C^l for any $l \geq 0$.

Exercise 2.2. (1) Show that for any $f \in L^{1,1}(\mathbb{R}^2)$, $f^2 \in L^1(\mathbb{R}^2)$, and moreover, show that the map $\Phi : L^{1,1}(\mathbb{R}^2) \to L^1(\mathbb{R}^2)$ sending f to f^2 defines a smooth map between the Banach spaces.

(2) Let $p > 2$. Show that for any $f \in L^{1,p}(\mathbb{R}^2)$, $f^2 \in L^{1,p}(\mathbb{R}^2)$, and $\Psi : L^{1,p}(\mathbb{R}^2) \to L^{1,p}(\mathbb{R}^2)$, $\Psi(f) = f^2$, is a smooth map between the Banach spaces.

Proposition 2.3. *Let $\sigma(k,p) := k - N/p > 0$ and let $H \in C^\infty(\mathbb{R})$. Suppose $D \subset \mathbb{R}^N$ is a bounded domain. Then for any $f \in L^{k,p}(D)$, the composition $H \circ f$ is also in $L^{k,p}(D)$, and moreover, the map $\Psi : L^{k,p}(D) \to L^{k,p}(D)$ defined by $\Psi(f) = H \circ f$ is a smooth map between the Banach spaces.*

Proof. First we show that for any $f \in L^{k,p}(D)$, $H \circ f \in L^{k,p}(D)$. We will only show that $H \circ f \in L^p(D)$ and $\partial_i(H \circ f) \in L^p(D)$, $\forall 1 \leq i \leq N$. The higher derivatives of $H \circ f$ can be done similarly and are left as exercises.

Since $\sigma(k,p) > 0$, the Morrey's embedding theorem gives $L^{k,p}(D) \hookrightarrow C^l(D)$ for some integer $l \geq 0$. In particular, we have $\sup_{x\in D} |f(x)| \leq C||f||_{k,p} \leq C_1 < \infty$ for some constant $C_1 > 0$. With this understood, $\sup_{x\in D} |H(f(x))| \leq \sup_{|t|\leq C_1} |H(t)| < \infty$, so that $H \circ f \in L^p(D)$. Similarly, $\partial_i(H \circ f) = H'(f)\partial_i f$, so that $||\partial_i(H \circ f)||_p \leq \sup_{|t|\leq C_1} |H'(t)| \cdot ||\partial_i f||_p$.

Next we show that Ψ is a smooth map. First we claim that $D\Psi(f) \in L(L^{k,p}(D), L^{k,p}(D))$ is given by the multiplication by $H' \circ f$. (We leave as an exercise to show that multiplication by $H' \circ f$ indeed defines a linear operator from $L^{k,p}(D)$ to itself, and moreover, its operator norm is bounded by a polynomial of degree at most k in $||f||_{k,p}$.) The point here is that if we write $\forall g \in L^{k,p}(D)$, $H(f+g) - H(f) - H'(f)\cdot g = R(g)$, then $||R(g)||_{k,p} \leq C||g||^2_{k,p}$. (For example, $R(g)$ satisfies that $\forall x \in D$, $|R(g(x))| \leq C|g(x)|^2$. From this we get $||R(g)||_p \leq C||g||^2_p$.) One can similarly show that $D^2\Psi(f) \in L(L^{k,p}(D) \times L^{k,p}(D), L^{k,p}(D))$ is given by $D^2\Psi(f)(g,h) = H^{(2)}(f)gh$. $\square$

Remark 2.4. Proposition 2.3 allows one to define the notion of locally $L^{k,p}$-maps from a smooth manifold of dimension N into another smooth manifold whenever $\sigma(k,p) = k - N/p > 0$. Note also that locally $L^{k,p}$-maps are continuous maps by the Morrey's embedding theorem (with the assumption that $\sigma(k,p) := k - N/p > 0$).

Definition 2.5. (1) (Banach manifolds modeled on a Banach space E) A topological space X is called a *smooth Banach manifold* if the following are satisfied: (i) X is Hausdorff, i.e., for any $x, y \in X$, there exist open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$; (ii) X is second countable, i.e., there exists a countable basis; (iii) there exists a smooth atlas $\mathcal{U} = \{(U,\phi)\}$, where $\{U\}$ is an open cover of X, and each $\phi : U \to E$ is a homeomorphism onto an open set $\phi(U) \subset E$, such that for any U_1, U_2, $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a smooth map (in fact a diffeomorphism). Elements of $\mathcal{U}$ are called smooth charts. Without loss of generality, one assumes $\mathcal{U}$ is maximal.

(2) (Tangent space) Let $x \in X$ be any point. Denote by A_x the set of smooth charts containing x. The tangent space at x, denoted by T_xX, is the quotient $A_x \times E/\sim$, where $((U,\phi), v) \sim ((V,\psi), w)$ if $v = D(\phi\circ\psi^{-1})_{\psi(x)}(w)$. We remark that as topological spaces T_xX is isomorphic to E, but T_xX does not have an intrinsic norm to make it into a Banach space, even though each smooth chart $(U,\phi) \in A_x$ gives rise to a specific identification of T_xX with E as Banach spaces.

(3) (Smooth maps between Banach manifolds) A map $f : X \to Y$ between two Banach manifolds is said to be *smooth at a point* $x \in X$ if there exists a smooth chart (U,ϕ) containing x and a smooth chart (V,ψ) containing $y = f(x) \in Y$, such that the composition $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is a smooth map at $\phi(x)$ between open sets of Banach spaces. A map $f : X \to Y$ is called *smooth* if it is smooth at every point in X.

(4) (Differential or linearization of a smooth map) Let X, Y be Banach manifolds modeled by Banach spaces E, F respectively. Let $f : X \to Y$ be a map which is smooth at $x \in X$, and let $y = f(x)$. The differential of f at x, denoted by Df_x, is the linear map $Df_x : T_xX \to T_yY$, which for $(U,\phi) \in A_x$, $(V,\psi) \in A_y$, Df_x is represented by the bounded linear operator $D(\psi \circ f \circ \phi^{-1})_{\phi(x)} \in L(E,F)$.

Definition 2.6. (1) (Banach bundles and fibrations) Let X be a Banach manifold and E be a Banach space. A *Banach bundle with typical fiber* E is a Banach manifold $\mathcal{E}$ together with a smooth, surjective map $\pi : \mathcal{E} \to M$ such that (i) for any $x \in X$, the fiber at x $\mathcal{E}_x := \pi^{-1}(x)$ is a topological vector space isomorphic to E; (ii) for any $x \in X$, there exists an open neighborhood U of x and a diffeomorphism of Banach manifolds $\Phi : \pi^{-1}(U) \to U \times E$ such that $\pi = \pi_1 \circ \Phi$ and $\Phi|_{\mathcal{E}_x} : \mathcal{E}_x \to E$ is a linear isomorphism. Here $\pi_1 : U \times E \to U$. Φ is called a *local trivialization* over U. If one replaces the Banach space E by an arbitrary Banach manifold, one gets the notion of a *Banach fibration*.

(2) (Smooth sections of a Banach bundle) Let $\mathcal{E} \to X$ be a Banach bundle over a Banach manifold. A smooth map $s : X \to \mathcal{E}$ is called a *smooth section* if $\pi \circ s = Id_X$.

Exercise 2.7. (Tangent bundle of a Banach manifold) Let X be a Banach manifold, and let $TX := \bigcup_{x\in X} T_xX$. Show that there is a natural Banach manifold structure on TX such that together with the map $\pi : TX \to X$, $\pi(x, v) = x$, $\forall x \in X, v \in T_xX$, TX is a Banach bundle over X.

Let E be a Banach space. Recall that a closed subspace $F \subset E$ is said to *split* E if there exists a closed subspace $F_1 \subset E$ such that $E = F \oplus F_1$. In this case, E is naturally isomorphic to the product $F \times F_1$ by the Closed Graph Theorem (cf. [15]).

Definition 2.8. (1) (Submersions, immersions, and embeddings) Let $f : X \to Y$ be a smooth map between Banach manifolds. (i) f is a *submersion* if for any $x \in X$, $Df_x : T_xX \to T_{f(x)}Y$ is surjective and $\ker Df_x$ splits T_xX. (ii) f is an *immersion* if for any $x \in X$, $Df_x : T_x \to T_{f(x)}Y$ is injective and its image splits $T_{f(x)}Y$. (iii) f is a *smooth embedding* if f is an immersion and a homeomorphism onto the image $f(X) \subset Y$ given with the subspace topology.

(2) (Embedded submanifolds) Let E be a Banach space and F be a closed subspace which splits E. Let X be a Banach manifold modeled on E. A subset $Y \subset X$ is called an *embedded submanifold modeled on* F if for any $y \in Y$, there exists a smooth chart (U, ϕ) of X containing y such that $\phi(Y \cap U) = \phi(U) \cap (F \times \{0\}) \subset E$. (Note that Y naturally becomes a Banach manifold and the inclusion map $Y \hookrightarrow X$ is naturally a smooth embedding.)

Next we list a few fundamental theorems about Banach manifolds.

Theorem 2.9. *(Banach contraction principle) Let $B_R \subset E$ be a (closed) ball of radius R in a Banach space, and let $T : B_R \to B_R$ be a self-map which satisfies the following contraction condition: there exists a $\lambda \in [0, 1)$ such that $||Tx - Ty|| \le \lambda||x - y||$ for any $x, y \in B_R$. Then there exists a unique $x_0 \in B_R$ such that $Tx_0 = x_0$.*

Proof. Let $x_1 \in B_R$ be the center of B_R and define inductively $x_{n+1} = Tx_n$, $\forall n \ge 1$. Then one easily sees that

$$||x_{n+1} - x_n|| = ||Tx_n - Tx_{n-1}|| \le \lambda||x_n - x_{n-1}|| \le \cdots \le \lambda^{n-1}||x_2 - x_1||.$$

Since $\lambda < 1$, it follows that $(x_n) \subset E$ is a Cauchy sequence. Let $x_0 = \lim_{n\to\infty} x_n$. Then $||x_0 - x_1|| = \lim_{n\to\infty} ||x_n - x_1|| \le R$, so that $x_0 \in B_R$. The relation $Tx_0 = x_0$ follows easily from the inductive relation $x_{n+1} = Tx_n$. For the uniqueness of x_0, suppose x_0' is another solution. Then $||x_0 - x_0'|| = ||Tx_0 - Tx_0'|| \le \lambda||x_0 - x_0'||$, which implies $x_0 - x_0' = 0$ since $\lambda \in [0, 1)$. □

The following theorem is a standard application of the Banach contraction principle.

Theorem 2.10. *(Inverse function theorem) Let $f : U \subset E \to F$ be a smooth map from an open set of a Banach space into another Banach space. If $Df(x_0) : E \to F$ is a bijection for some $x_0 \in U$, then there exist neighborhood U_0 of x_0 and V_0 of $y_0 := f(x_0)$, such that $f|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Theorem 2.11. *(Submersion theorem) Let $f : X \to Y$ be a submersion between two Banach manifolds. Then for any $y \in Y$, the preimage $f^{-1}(y) := \{x \in X | f(x) = y\} \subset X$ is an embedded submanifold of X.*

Proof. Suppose X, Y are modeled on Banach spaces E, F respectively. Let $x \in f^{-1}(y)$ be any point. we pick a smooth chart (U, ϕ) of X containing x and a smooth chart (V, ψ) of Y containing y. Note that this gives rise to an identification of T_xX with E and T_yY with F respectively. Moreover, let $E_1 \subset E$ be the closed subspace which corresponds to $\ker Df(x)$ under the identification, then by the definition of submersions, E_1 splits E. In particular, there exists a bounded linear map $\pi : E \to E_1$, the projection onto E_1.

With the preceding understood, consider the smooth map $g : \phi(U) \subset E \to F \times E_1$, where $g(u) = \psi \circ f \circ \phi^{-1}(u) + \pi(u)$, $\forall u \in \phi(U)$. Without loss of generality, we assume $\phi(x) = 0$ and $\psi(y) = 0$. Then $Dg(0) = D(\psi \circ f \circ \phi^{-1})(0) + \pi$, which is easisly seen bijective. By the inverse function theorem, g is a local diffeomorphism. Let $\Phi := g \circ \phi$. Then Φ defines a smooth chart over a neighborhood U_0 of x, and moreover, $\Phi(f^{-1}(y) \cap U_0) = \Phi(U_0) \cap (\{0\} \times E_1)$. This proves that $f^{-1}(y)$ is an embedded submanifold. $\square$

Example 2.12. Let M, N be compact closed, oriented, smooth manifolds of dimension m, n respectively. Consider the space $X := L^{k,p}(M; N)$ of locally $L^{k,p}$-maps from M into N, where $\sigma(k, p) = k - m/p > 0$. We claim X is naturally a Banach manifold, and moreover, at each $u \in X$, the tangent space T_uX can be identified with the Sobolev space $L^{k,p}(u^*TN)$, where $u^*TN \to M$ is the pull-back of TN via u. (Suppose $X = \sqcup X_\alpha$ are the components of X. Then for each X_α, there is a unique isomorphism class of pull-back bundles $E_\alpha = u^*TN$, $\forall u \in X_\alpha$, and X_α is modeled on the Banach space $L^{k,p}(E_\alpha)$.)

To see that X is a Banach manifold, we first give a topology to X as follows. Pick a smooth embedding $N \hookrightarrow \mathbb{R}^l$ for some large $l \geq 0$. Then we can regard X as the subset of $L^{k,p}(M; \mathbb{R}^l)$ (which is a Banach space) consisting of those u such that $u(M) \subset N$. We give X the subspace topology, which is naturally Hausdorff and second countable. It remains to prove the existence of a smooth atlas on X.

We give N the induced metric from $\mathbb{R}^l$. Then each u^*TN has a metric and a connection which is the pull-back of the Levi-Civita connection on N. We also fix a metric on M. With these choices of data, we can define a norm $||\cdot||_{k,p}$ on $L^{k,p}(u^*TN)$. With this understood, since M is compact, there exists an $\epsilon_0 = \epsilon_0(u) > 0$, such that if we let $B_{\epsilon_0}(u) \subset L^{k,p}(u^*TN)$ be the ball of radius ϵ_0 (with respect to the norm $||\cdot||_{k,p}$), then the following map $\Phi_u : B_{\epsilon_0}(u) \to X$, is defined, where $\Phi_u(\xi)(x) = \exp_{u(x)} \xi(x)$, $\forall x \in M$. Here we use the fact that $\sup_x |\xi(x)| \leq C||\xi||_{k,p}$ by the Morrey's embedding theorem. The fact is that $\{(\Phi_u(B_{\epsilon_0}(u)), \Phi_u^{-1})\}$ form a smooth atlas on X, i.e., $\Phi_u : B_{\epsilon_0}(u) \to X$ is a homeomorphism onto its image, and $\Phi_u^{-1} \circ \Phi_v$, $\forall u, v \in X$, is smooth whenever it is defined. The formal follows from properties of the exponential map and the latter uses Proposition 2.3. We leave the details as an exercise.

We end this section with a brief review on some fundamental properties of Fredholm operators. Let E, F be Banach spaces, and let $L(E, F)$ be the Banach space of bounded linear operators given with the operator norm. Recall that (cf. Def. 1.76) a $L \in$

$L(E,F)$ is called a Fredholm operator if $\dim\ker L < \infty$, Im $L \subset F$ is a closed subspace, and coker $L := F/\text{Im } L$ has finite dimension. The index of L is defined to be

$$\text{Index } L = \dim\ker L - \dim\text{coker } L.$$

We denote by Fred (E,F) the subset of Fredholm operators.

The proof of the following result can be found, for example, in [7, 10, 12].

Theorem 2.13. *Suppose $L \in$ Fred (E,F). Then the following are true.*

(1) There exists an $\epsilon > 0$, such that if $P \in L(E,F)$ satisfies $||P|| < \epsilon$, then $L+P \in$ Fred (E,F), and moreover, Index $(L+P) =$ Index L.

(2) Suppose $K \in L(E,F)$ is a compact operator. Then $L+K \in$ Fred (E,F), and moreover, Index $(L+K) =$ Index L.

Remark 2.14. (1) Theorem 2.13(1) shows that Fred (E,F) is an open subset of $L(E,F)$ and the index of a Fredholm operator is a locally constant function.

(2) A basic example of Theorem 2.13(2): Let L_0 be an elliptic p.d.o. on a compact manifold and $L = L_0 + P$ be a p.d.o. where P has order $\leq k-1$. For any integer $m \geq 0$ and $p > 1$, the operator $L : L^{m+k,p} \to L^{m,p}$ is a compact perturbation of $L_0 : L^{m+k,p} \to L^{m,p}$ because by Rellich-Kondrachov, $P : L^{m+k,p} \to L^{m,p}$ is compact.

2.2. **Moduli space of nonlinear elliptic equations.** Recall that the idea of studying an elliptic p.d.o. $L : C^\infty(E) \to C^\infty(F)$ is to take the completion of $C^\infty(E), C^\infty(F)$ under certain Sobolev norms and study the corresponding operator between the Sobolev spaces. In particular, one derives the Fredholm properties of the operator. Note that in this approach, the apriori estimates and elliptic regularity have played a fundamental role. Similarly, one can adapt this idea to study nonlinear elliptic equations in the framework of Banach manifolds (which is the so-called (non-linear) Fredholm theory).

Example 2.15. (Pseudoholomorphic curves) Let (M,J) be a compact closed almost complex manifold of dimension $2n$ (here J is a smooth endomorphism of TM such that $J^2 = -Id$), and let (Σ, j_0) be a Riemann surface of genus 0, where j_0 denotes the unique complex structure on Σ. A smooth map $u : \Sigma \to M$ is called a *J-holomorphic curve* if the following equation is satisfied

$$J \circ du = du \circ j_0.$$

One is interested in the set of all such maps, i.e., the moduli space $\mathcal{M}$ of J-holomorphic curves. To this end, we let $\mathcal{B} = C^\infty(\Sigma; M)$ be the space of all smooth maps from Σ to M, and consider the vector bundle $\pi : \mathcal{E} \to \mathcal{B}$, whose fiber $\mathcal{E}_u := \pi^{-1}(u)$ at $u \in \mathcal{B}$ is $\Omega^{0,1}(u^*TM)$, the space of smooth $(0,1)$-forms on Σ with values in the complex vector bundle (u^*TM, u^*J). Then $\mathcal{M}$ may be regarded as the zero set of a section $s : \mathcal{B} \to \mathcal{E}$, $s : u \mapsto (u, \bar\partial_J(u))$, where

$$\bar\partial_J(u) = \frac{1}{2}(du + J \circ du \circ j_0), \ \ \forall u \in \mathcal{B}.$$

(Note that $\bar\partial_J(u) \in \Omega^{0,1}(u^*TM)$ exactly means that $\bar\partial_J \circ j_0 = -J \circ \bar\partial_J$.)

In order to put this problem in the framework of Banach manifolds, we choose $k \in \mathbb{Z}^+$, $1 < p < \infty$, such that $\sigma(k,p) := k - 2/p > 0$, and consider the Banach manifold $\mathcal{B}_{k,p}$ of locally $L^{k,p}$-maps from Σ to M, and the infinite dimensional vector bundle $\pi : \mathcal{E}_{k,p} \to \mathcal{B}_{k,p}$ whose fiber at u is the Sobolev space $L^{k-1,p}(\Lambda^{0,1} \otimes u^*TM)$. We claim that $\pi : \mathcal{E}_{k,p} \to \mathcal{B}_{k,p}$ is a Banach bundle, and that $s = (Id, \bar{\partial}_J)$ defines a smooth section of $\pi : \mathcal{E}_{k,p} \to \mathcal{B}_{k,p}$.

To see that $\pi : \mathcal{E}_{k,p} \to \mathcal{B}_{k,p}$ is a Banach bundle, recall (from Example 2.12) that $\mathcal{B}_{k,p}$ is given a smooth atlas $\{(\Phi_u(B_{\epsilon_0}(u)), \Phi_u^{-1})\}$, where $B_{\epsilon_0}(u) \subset L^{k,p}(u^*TM)$ is the ball of radius ϵ_0, and $\Phi_u : B_{\epsilon_0}(u) \to X$ is defined by $\Phi_u(\xi)(z) = \exp_{u(z)} \xi(z)$, $\forall z \in \Sigma$. Over each $\Phi_u(B_{\epsilon_0}(u))$, we identify $\pi^{-1}(\Phi_u(B_{\epsilon_0}(u))) \subset \mathcal{E}_{k,p}$ with the product $\Phi_u(B_{\epsilon_0}(u)) \times L^{k-1,p}(\Lambda^{0,1} \otimes u^*TM)$,

$$\Psi_u : \pi^{-1}(\Phi_u(B_{\epsilon_0}(u))) \to \Phi_u(B_{\epsilon_0}(u)) \times L^{k-1,p}(\Lambda^{0,1} \otimes u^*TM),$$

where the fiber over $\Phi_u(\xi)$, $L^{k-1,p}(\Lambda^{0,1} \otimes (\Phi_u(\xi))^*TM)$, is identified with $L^{k-1,p}(\Lambda^{0,1} \otimes u^*TM)$ by the isomorphism of Sobolev spaces induced by the isomorphism of the corresponding bundles $(\Phi_u(\xi))^*TM \to u^*TM$ given by the parallel transport along the geodesics $\exp(t\xi(z))$, $t \in [0,1]$, $z \in \Sigma$. The transition maps for the trivializations over different balls are smooth, which shows that $\pi : \mathcal{E}_{k,p} \to \mathcal{B}_{k,p}$ is a Banach bundle.

To see that $s = (Id, \bar{\partial}_J)$ defines a smooth section of $\pi : \mathcal{E}_{k,p} \to \mathcal{B}_{k,p}$, we need to show that the local representatives of s over the smooth charts:

$$(\Phi_u^{-1}, Id) \circ \Psi_u \circ (Id, \bar{\partial}_J) \circ \Phi_u : B_{\epsilon_0}(u) \to B_{\epsilon_0}(u) \times L^{k-1,p}(\Lambda^{0,1} \otimes u^*TM)$$

are smooth maps of Banach spaces. In order to do this, let's compute $\bar{\partial}_J$ in local coordinates. Let $z = s + it$ be a local holomorphic coordinate over a neighborhood D of a point $z_0 \in \Sigma$, and let $\phi : U \to \mathbb{R}^{2n}$ be a local smooth chart of M near $u(z_0)$. We let $v(z) := \phi \circ u(z) : D \to \mathbb{R}^{2n}$ be the local representative of u in these local coordinates. Then

$$\bar{\partial}_J(v) = \frac{1}{2}(\partial_s v + J(v)\partial_t v)ds + \frac{1}{2}(\partial_t v - J(v)\partial_s v)dt.$$

Now if we choose D small enough, an element $\xi \in B_{\epsilon_0}(u)$ over D may be written as $(z, \eta(z))$ for an $\eta \in L^{k,p}(D; \mathbb{R}^{2n})$. Moreover, if we choose ϕ to be the inverse of the exponential map, then $\xi \mapsto \bar{\partial}_J \circ \Phi_u(\xi)$ may be expressed locally over D as

$$\eta \mapsto \frac{1}{2}(\partial_s \eta + J(\eta)\partial_t \eta)ds + \frac{1}{2}(\partial_t \eta - J(\eta)\partial_s \eta)dt.$$

Since $\eta \in L^{k,p}(D; \mathbb{R}^{2n})$, we have both $\partial_s \eta, \partial_t \eta \in L^{k-1,p}(D; \mathbb{R}^{2n})$. Moreover, by Prop. 2.3, $\eta \mapsto J(\eta)$ is a smooth map from $L^{k,p}(D; \mathbb{R}^{2n})$ to itself. Hence the map above on η is a smooth map from $L^{k,p}(D; \mathbb{R}^{2n})$ to $L^{k-1,p}(D; \mathbb{R}^{2n})$ if the product $L^{k,p}(D; \mathbb{R}^{2n}) \times L^{k-1,p}(D; \mathbb{R}^{2n}) \to L^{k-1,p}(D; \mathbb{R}^{2n})$ is smooth. If $k = 1$, this is certainly true by Morrey's embedding theorem $L^{k,p} \hookrightarrow C^0$ (by assumption $\sigma(k,p) > 0$). If $k > 1$, one may further assume that $\sigma(k-1,p) > 0$ to ensure this. Finally, locally over D, the section s may be expressed as $\xi \mapsto (\xi, P(\bar{\partial}_J \circ \Phi_u(\xi)))$ where P is a certain isomorphism of the Sobolev space $L^{k-1,p}(D; \mathbb{R}^{2n})$ induced by an isomorphism of $\mathbb{R}^{2n}$. This last map P corresponds to Ψ_u and is resulted from the parallel transport. This proves that s is a smooth section of the Banach bundle $\pi : \mathcal{E}_{k,p} \to \mathcal{B}_{k,p}$, at least under assumption that $\sigma(k,p) > 0$ is sufficiently large.

In fact, the section s is also what we will call a *Fredholm section*.

Definition 2.16. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a Banach bundle. A smooth section $s : \mathcal{B} \to \mathcal{E}$ is called Fredholm if for any $x \in s^{-1}(0)$ in the zero set, when writing s locally as a gragh of f, the derivative $Df(x) : T_x\mathcal{B} \to \mathcal{E}_x$ is a Fredholm operator of Banach spaces.

Back to Example 2.15, in order to see that s is a Fredholm section, we will look at the derivative of the following map, denoted by D_η, at η:

$$\eta \mapsto \frac{1}{2}(\partial_s\eta + J(\eta)\partial_t\eta)ds + \frac{1}{2}(\partial_t\eta - J(\eta)\partial_s\eta)dt.$$

Let $\tau \in L^{k,p}(D; \mathbb{R}^{2n})$. Here τ, as a $L^{k,p}$-section of the pull-back bundle u^*TM over D, should be thought of as a $L^{k,p}$-vector field of M along the image $u(D) \subset M$. With this understood, differentiating the above map in the direction of τ, we obtain the derivative D_η which is

$$D_\eta(\tau) = \frac{1}{2}(\partial_s\tau + J(\eta)\partial_t\tau + (\partial_\tau J)(\eta)\partial_t\eta)ds + \frac{1}{2}(\partial_t\tau - J(\eta)\partial_s\tau - (\partial_\tau J)(\eta)\partial_s\eta)dt.$$

If we set $\bar\partial_{J,u}v := \frac{1}{2}(dv + J(u) \circ dv \circ j_0)$ and $\partial_{J,u}v := \frac{1}{2}(dv - J(u) \circ dv \circ j_0)$, then u is a J-holomorphic curve means that $\bar\partial_{J,u}u = 0$, hence $\partial_{J,u}u = -J(u) \circ du \circ j_0$. With these relations, we can write D_η as

$$D_\eta\tau = \bar\partial_{J,\eta}\tau - \frac{1}{2}J(\eta)(\partial_\tau J)(\eta)\partial_{J,\eta}\eta,$$

assuming $\eta \in s^{-1}(0)$. At this point, we need the following regularity result.

Theorem 2.17. *(Regularity of J-holomorphic curves, cf.* [10]*, Thm B.4.1.) Fix $l \geq 1$ and $p > 2$. If u is of $L^{1,p}$-class and satisfies the J-holomorphic curve equation $\bar\partial_J u = 0$ for an almost complex structure J of C^l-class, then u is of $L^{l+1,p}$ class. In particular, if J is smooth, so is u.*

With this regularity result, we see that D_η is a first order p.d.o. (with smooth coefficients !). Moreover, one can check that D_η is a generalized Cauchy-Riemann operator, henec elliptic. By Theorem 1.75, s is a Fredholm section. Note also that if we set $\mathcal{M}_{k,p} := s^{-1}(0)$ to be the zero set of the section $s : \mathcal{B}_{k,p} \to \mathcal{E}_{k,p}$, then $\mathcal{M}_{k,p} = \mathcal{M}$ by Theorem 2.17, which is independent of k, p as long as $k \geq 1$ and $p > 2$.

Exercise 2.18. Prove the regularity of J-holomorphic curves under the following stronger assumption: if u is of $L^{k,p}$-class and J-holomorphic for a smooth J, where $\sigma(k,p) := k - 2/p > 1$, then u is smooth.

Example 2.19. (Seiberg-Witten equations, cf. [11]) Let (X, g) be a compact closed, oriented 4-dimensional Riemannian manifold. A $Spin^{\mathbb{C}}$-structure on (X, g) is given by a pair of rank 2 Hermitian vector bundles, $\mathbb{S}^+$, $\mathbb{S}^-$, with the same determinant line bundle $L := \det \mathbb{S}^+ = \det \mathbb{S}^-$. Furthermore, the bundle of traceless endomorphisms of $\mathbb{S}^+$ is identified via the Clifford multiplication with the bundle of complex valued self-dual 2-forms on X. This being said, if we let ψ^* be the dual of $\psi \in \mathbb{S}^+$ via the Hermitian metric, then $q(\psi) := \psi \otimes \psi^* - \frac{1}{2}|\psi|^2 Id$ is a traceless endomorphism of $\mathbb{S}^{+1}$, hence defines a self-dual 2-form on X, which can be easily seen to be purely imaginary valued. On the other hand, let A be any $U(1)$-connection on the Hermitian line bundle

L. Together with the Levi-Civita connection on (X, g), A determines a Dirac operator $D_A : C^\infty(\mathbb{S}^+) \to C^\infty(\mathbb{S}^-)$ (a first order elliptic p.d.o.).

The Seiberg-Witten equations are the following equations for a pair (A, ψ), where A is a smooth $U(1)$-connection on L and $\psi \in C^\infty(\mathbb{S}^+)$ (called a spinor field):

$$\begin{cases} F_A^+ = q(\psi) \\ D_A\psi = 0 \end{cases}$$

Here F_A^+ denotes the self-dual part of the curvature F_A of A, which is a purely imaginary valued self-dual 2-form. The space of solutions of the Seiberg-Witten equations carries nontrivial information about the smooth structure of X (which is used to define the so-called Seiberg-Witten invariants of X). We shall explain below how to study the Seiberg-Witten solution space in the framework of Banach manifolds.

The first observation is that the Seiberg-Witten equations are not elliptic. One way to understand this is that there is an infinite dimensional group of gauge transformations acting on the space of pairs (A, ψ) such that the Seiberg-Witten equations are invariant under gauge transformations. More precisely, let $\mathcal{G}$ be the space of smooth circle valued functions on X, which is an infinite dimensional abelian Lie group under the pointwise multiplication. For any $\sigma \in \mathcal{G}$, $\sigma^{-1}d\sigma$ is a purely imaginary valued 1-form. The action of $\mathcal{G}$ is given by $\sigma \cdot (A, \psi) = (A - \sigma^{-1}d\sigma, \sigma\psi)$. With this understood, if we define a map $F : (A, \psi) \mapsto (F_A^+ - q(\psi), D_A\psi)$ and let $\mathcal{G}$ act on the first component of the target trivially and on the second component by pointwise multiplication, then F is a $\mathcal{G}$-equivariant map.

Even though the Seiberg-Witten equations are not elliptic, they are elliptic modulo the action of $\mathcal{G}$. This suggests one should consider the Seiberg-Witten equations as defined on the quotient space of the action of $\mathcal{G}$.

With the preceding understood, the Banach manifold setup goes as follows. Let

$$\mathcal{C} := \{(A, \psi) | A \text{ is of } L^{2,2} \text{ class and } \psi \in L^{2,2}(\mathbb{S}^+)\},$$

and $\tilde{\mathcal{C}} := L^{1,2}(\Lambda^{2,+} \otimes i\mathbb{R}) \times L^{1,2}(\mathbb{S}^-)$. Then by the Sobolev embedding theorem and Hölder inequality that $F : \mathcal{C} \to \tilde{\mathcal{C}}$, $(A, \psi) \mapsto (F_A^+ - q(\psi), D_A\psi)$, is a smooth map. If we consider the $L^{3,2}$-completion of the group of gauge transformations, which is still denoted by $\mathcal{G}$ for simplicity, then $\mathcal{G}$ acts on $\mathcal{C}$ and $\tilde{\mathcal{C}}$ smoothly. Set $\mathcal{B} := \mathcal{C}/\mathcal{G}$ and let $\pi : \mathcal{E} \to \mathcal{B}$ be the bundle which is the descendant of the trivial bundle $\mathcal{C} \times \tilde{\mathcal{C}} \to \mathcal{C}$ under the action of $\mathcal{G}$, then F descends to a section $\tilde{F} : \mathcal{B} \to \mathcal{E}$. Let $\mathcal{B}^* \subset \mathcal{B}$ be the subspace which consists of gauge equivalence classes of (A, ψ) where ψ is not identically zero.

Proposition 2.20. *(cf.* [11]*) $\mathcal{B}^*$ is naturally a Banach manifold and $\pi : \mathcal{E} \to \mathcal{B}^*$ a Banach bundle. Moreover, $\tilde{F} : \mathcal{B}^* \to \mathcal{E}$ is a smooth, Fredholm section.*

In the rest of this section, we shall present a few basic results concerning the zero set of a smooth, Fredholm section of a Banach bundle. First, we observe

Lemma 2.21. *Let $E_1 \subset E$ be a closed subspace of a Banach space. Then E_1 splits E if one of the following conditions are satisfied: (1) $\dim(E/E_1) < \infty$, or (2) $\dim E_1 < \infty$.*

Proof. Consider case (1) that $\dim(E/E_1) = n < \infty$. Let $e_1', \cdots, e_n'$ be a basis of E/E_1 and let $e_1, \cdots, e_n$ be the lift to E, then $E_2 :=$ span $(e_1, \cdots, e_n)$ is a n-dimensional closed subspace of E, and $E = E_1 \oplus E_2$.

Consider case(2) that $\dim E_1 = n < \infty$. Let $e_1, \cdots, e_n$ be a basis of E_1 and let $f_1, \cdots, f_n \in E^*$ be its dual, i.e., $f_i(e_j) = \delta_{ij}$. We define a bounded linear operator $P : E \to E_1$ by $Pu = \sum_{i=1}^n f_i(u) \cdot e_i$. (Note that $||P|| \leq \sum_{i=1}^n ||f_i|| \cdot ||e_i||$.) Then one can check easily that P is a projection, i.e., $P^2 = P$. Let $E_2 := \ker P$, then E_2 is a closed subspace and $E = E_1 \oplus E_2$. □

Let E, F be Banach spaces and $U \subset E$ be an open set. A smooth map $f : U \to F$ is called a *Fredholm map* if for any $x \in U$, $Df(x) : E \to F$ is a Fredholm operator. A smooth map between Banach manifolds is Fredholm if its local representatives are Fredholm. Fredholm maps admit a useful finite dimensional reduction called *Kuranishi model*.

Theorem 2.22. *(Kuranishi Model) Let $f : U \subset E \to F$ be a Fredholm map such that $f(0) = 0$. Set $D := Df(0) : E \to F$. Then there exist an open neighborhood U_0 of 0, a diffeomorphism $g : U_0 \to U_0$ and a smooth map $f_0 : U_0 \to$ coker D, such that*

$$f \circ g(x) = f_0(x) + Dx, \ \ \forall x \in U_0,$$

where $g(0) = 0$, $Dg(0) = Id_E$, $f_0(0) = 0$, and $Df_0(0) = 0$. In particular, $U_0 \cap f^{-1}(0)$ is identified under g with the zero set of a smooth map between finite dimensional spaces, $(f_0|_{\ker D})^{-1}(0)$.

Proof. By Lemma 2.21, $\ker D$ splits E and $\mathrm{Im}\, D$ splits F, so that we may write $E = E_1 \times \ker D$ and $F = F_1 \times \mathrm{coker}\ D$, where $D|_{E_1} : E_1 \to F_1$ is a bijection. We define a bounded linear operator $T : F \to E_1 \subset E$, which is the projection onto F_1 followed by D^{-1}.

Consider the smooth map $\phi : U \to E$, where $\phi(x) = x + T(f(x) - Dx)$. Then $D\phi(0) = Id_E$ so that by the inverse function theorem, there exists an open neighborhood U_0 of 0 such that ϕ is a diffeomorphism from U_0 onto itself. We set $g := \phi^{-1}$, $f_0 := (Id_F - DT) \circ f \circ g : U_0 \to \mathrm{coker}\ D$ (note that by definition of T, $Id_F - DT$ is the projection onto coker D). Then one can check that $D\phi(x) = DTf(x)$, so that $D = DT \circ (f \circ g)$. This gives

$$f \circ g = (Id_F - DT) \circ (f \circ g) + D = f_0 + D.$$

□

Note that in the above theorem, if $D = Df(0)$ is surjective, then coker $D = 0$ and $U_0 \cap f^{-1}(0)$ is identified under g with $U_0 \cap \ker D$.

Definition 2.23. Let $s : \mathcal{B} \to \mathcal{E}$ be a smooth Fredholm section of a Banach bundle $\pi : \mathcal{E} \to \mathcal{B}$. We say s is *transverse to the zero section* if, when writing s locally as a gragh of f, the derivative $Df(x) : T_x\mathcal{B} \to \mathcal{E}_x$ is surjective for all $x \in s^{-1}(0)$.

Theorem 2.24. *Let $s : \mathcal{B} \to \mathcal{E}$ be a smooth Fredholm section of a Banach bundle $\pi : \mathcal{E} \to \mathcal{B}$ which is transverse to the zero section. Then the zero set $\mathcal{M} = s^{-1}(0)$ is a smooth, finite dimensional submanifold of $\mathcal{B}$, where for any $x \in \mathcal{M}$, the dimension of $\mathcal{M}$ at x is given by the index of $Df(x)$. (Here s is the graph of f near x.)*

2.3. **The Sard-Smale theorem and transversality.** The nonlinear elliptic problems we encounter often naturally come in families. For example, the J-holomorphic curve equation (cf. Example 2.15) depends on the choice of almost complex structure J, and the Seiberg-Witten equations (cf. Example 2.19) depend on the choice of the Riemannian metric g. On the other hand, in order to achieve transversality, one often consider perturbed equations. For instance, in the Seiberg-Witten theory, one adds a perturbation term η to the right hand side of the first equation in the Seiberg-Witten equations. With this understood, one is led to the consideration of the following problem: $s : \mathcal{B} \to \mathcal{E}$ is a smooth section of a Banach bundle $\pi : \mathcal{E} \to \mathcal{B}$, and there is a Banach manifold Λ with a Banach fibration $pr : \mathcal{B} \to \Lambda$ such that for each $\lambda \in \Lambda$, if we let $\mathcal{B}_\lambda := pr^{-1}(\lambda)$, $\mathcal{E}_\lambda := \mathcal{E}|_{\mathcal{B}_\lambda}$, and $s_\lambda := s|_{\mathcal{B}_\lambda}$, then s_λ is a Fredholm section. In the J-holomorphic curve example, Λ will be a certain Banach completion of the space of J's, and in the Seiberg-Witten example, Λ will be a certain Banach completion of the space of pairs (g, η). The purpose of this section is to present some techniques, which in particular allow us to show that for a generic choice of $\lambda \in \Lambda$, s_λ is transverse to the zero section. The key ingredient of this approach is the so-called Sard-Smale theorem, a generalization by S. Smale of the classical Sard's theorem concerning critical values of a smooth function between finite dimensional spaces to the infinite dimensional setting (cf. [14]).

Theorem 2.25. *(Sard's Theorem) Let U be an open set of $\mathbb{R}^p$ and $f : U \to \mathbb{R}^q$ be a C^s-map where $s > \max(p-q, 0)$. Then the set of critical values of f in $\mathbb{R}^q$ has measure zero.*

Theorem 2.26. *(The Baire-Hausdorff Theorem, cf.* [15]*) A nonempty complete metric space can not be expressed as the union of a countable number of non-dense subsets. (A subset is called non-dense if its closure does not contain any nonempty open subsets.)*

Definition 2.27. A subset of a topological space is said of *Baire's second category* if it can be expressed as the intersection of a countable number of open, dense subsets.

Note that the Baire-Hausdorff theorem implies that a subset of Baire's second category of a (nonempty) Banach manifold is everywhere dense; in particular, it is nonempty.

Theorem 2.28. *(Sard-Smale) Let $f : X \to Y$ be a smooth, Fredholm map between Banach manifolds. Then the set of regular values*

$$Y_{reg}(f) := \{y \in Y | f^{-1}(y) = \emptyset, \text{ or } Df(x) \text{ is surjective}, \forall x \in f^{-1}(y)\}$$

is a subset of Y of Baire's second category. (In particular, $Y_{reg}(f) \neq \emptyset$.)

Proof. For every point $x \in X$, there is a local Kuranishi model of f, where $f : U_x \to Y$, such that $f = f_0 + D$ for a smooth map $f_0 : U_x \to \mathbb{R}^q$ and a linear map D whose image is complement to $\mathbb{R}^q$. Since X is second countable, we can cover X by a countable collection of such U_x's. In fact, the following can be arranged: there is a countable set of open subsets U_i, such that (1) $X = \cup_i V_i$ where $V_i \subset U_i$ is a closed ball, (2) on each U_i, $f = f_0 + D$ as we described above. To see this, for each $x \in X$ we fix a local

Kuranishi model as above and also choose a closed ball $V_x \subset U_x$. Now let $\mathcal{B} := \{B_i\}$ be the subset of the countable basis where each B_i is contained in the interior of one of the V_x's. It follows easily that $\mathcal{B}$ covers X. Now for each $B_i \in \mathcal{B}$, pick a V_x whose interior contains it and call it V_i. Then $X = \cup_i V_i$. Name the U_x to be U_i.

We set

$$Y_{reg}(f, V_i) := \{y \in Y | f^{-1}(y) \cap V_i = \emptyset, \text{ or } Df(x) \text{ is surjective}, \forall x \in f^{-1}(y) \cap V_i\},$$

then $Y_{reg}(f) = \cap_i Y_{reg}(f, V_i)$ because $X = \cup_i V_i$. So it suffices to show that $Y_{reg}(f, V_i)$ is open and dense.

Let's consider $Y_{reg}(f, V_i)$. Let $W_i \times E_i$ be an open set in Y where $f_0(V_i) \subset W_i \subset \mathbb{R}^q$ and $D(V_i) \subset E_i$. Let $K_i := \ker D \cap V_i$, and $F_i \subset V_i$ such that $D^{-1} : E_i \to F_i$. Without loss of generality, assume $V_i = K_i \times F_i$. Then one can check easily that for any $e \in E_i$, $(w, e) \in Y$ lies in $Y_{reg}(f, V_i)$ if and only if w is not a critical value of $f_0|_{K_i \times \{D^{-1}(e)\}}$. By Sard's theorem, $W_i \setminus \{$ critical values of $f_0|_{K_i \times \{D^{-1}(e)\}}\}$ is dense. It follows easily that $Y_{reg}(f, V_i)$ is dense.

To see $Y_{reg}(f, V_i)$ is open, let $(w_n, e_n) \in W_i \times E_i$ be a sequence of critical values of f, and let $(k_n, f_n) \in K_i \times F_i$ be the corresponding critical points. If (w_n, e_n) converges, then so does $f_n = D^{-1}(e_n)$, and since K_i is compact, k_n also converges after passing to a subsequence. Hence a subsequence of (k_n, f_n) converges to a critical point of f, which lies in V_i because V_i is closed. This shows that the complement of $Y_{reg}(f, V_i)$ is closed, hence it is open. □

With these preparations, let us consider a given family of Fredholm sections $s : \mathcal{B} \to \mathcal{E}$, $s = \{s_\lambda\}$, parametrized by $\lambda \in \Lambda$, where $pr : \mathcal{B} \to \Lambda$ is a Banach fibration.

Theorem 2.29. *Suppose $s : \mathcal{B} \to \mathcal{E}$ is transverse to the zero section. Then (1) $\mathcal{M} := s^{-1}(0)$ is an embedded submanifold of $\mathcal{B}$ such that $pr|_{\mathcal{M}} : \mathcal{M} \to \Lambda$ is a Fredholm map; (2) there exists a subset $\Lambda_{reg} \subset \Lambda$ of Baire's second category, such that for any $\lambda \in \Lambda_0$, s_λ is transverse to the zero section, in particular, $\mathcal{M}_\lambda := s_\lambda^{-1}(0)$ is a smooth finite dimensional submanifold of $\mathcal{B}_\lambda$.*

Proof. Suppose $x_0 \in \mathcal{M}$ be any point, and let $\lambda_0 = pr(x_0)$ be the image in Λ under the map pr. Since pr is a fibration, at least locally we can write a neighborhood of x_0 in $\mathcal{B}$ as a product $U \times V$, where U is a neighborhood of x_0 in $\mathcal{B}_{\lambda_0}$ and V is a neighborhood of λ_0 in Λ. Suppose s is given by the graph of f over $U \times V$, and let f_λ be the restriction of f to $\mathcal{B}_\lambda$. Set $D := Df(x_0)$, $D_1 = Df_{\lambda_0}(x_0)$. Note that we can write $D = D_1 + D_2$ by the decomposition $T_{x_0}\mathcal{B} = T_{x_0}\mathcal{B}_{\lambda_0} \times T_{\lambda_0}\Lambda$. Set $K := \ker D$.

Lemma 2.30. *The kernel of $D(pr)(x_0)|_K : K \to T_{\lambda_0}\Lambda$ is given by the subspace $\ker D_1 \subset K$, and the cokernel of $D(pr)(x_0)|_K$ is identified via D_2 to the coker D_1. Moreover, K splits $T_{x_0}\mathcal{B}$.*

Proof. Let $u \in K$. We write $u = (v, w)$ according to the decomposition $T_{x_0}\mathcal{B} = T_{x_0}\mathcal{B}_{\lambda_0} \times T_{\lambda_0}\Lambda$. Then $D_1 v + D_2 w = 0$. Now u is sent to w under $D(pr)(x_0)|_K$, so u lies in the kernel of $D(pr)(x_0)|_K$ iff $w = 0$, which means $u = (v, 0)$ with $v \in \ker D_1$. This proves our claim that the kernel of $D(pr)(x_0)|_K$ is given by the subspace $\ker D_1 \subset K$. Concerning the cokernel of $D(pr)(x_0)|_K$, note that the above argument also shows that $w \in T_{\lambda_0}\Lambda$ lies in the image of $D(pr)(x_0)|_K$ iff there exists a $v \in T_{x_0}\mathcal{B}_{\lambda_0}$ such that

$D_1v + D_2w = 0$, which implies that D_2 induces an injective map from the cokernel of $D(pr)(x_0)|_K$ to coker D_1. To see this map is also surjective, we recall the assumption that s is transverse to the zero section, which means that D is surjective. Therefore for any $\xi \in \mathcal{E}_{x_0}$, there is a $u = (v, w)$ such that $\xi = Du = D_1v + D_2w$. This shows that the class of w in the cokernel of $D(pr)(x_0)|_K$ is sent to the class of ξ in coker D_1 under D_2, which is exactly the surjectivity we claimed. This proves that the cokernel of $D(pr)(x_0)|_K$ is identified via D_2 to the coker D_1.

It remains to show that K splits $T_{x_0}\mathcal{B}$. We let $E := D(pr)(x_0)^{-1}(\text{Im } D(pr)(x_0)|_K)$. Since $D(pr)(x_0)$ is surjective and the cokernel of $D(pr)(x_0)|_K$ is finite dimensional, $T_{x_0}\mathcal{B}/E$ is finite dimensional, hence by Lemma 2.21(1), E splits $T_{x_0}\mathcal{B}$. Note that $K \subset E$, so it suffices to show that K splits E. To see this, note that by Lemma 2.21(2), $\ker D_1$ splits $T_{x_0}\mathcal{B}_{\lambda_0}$ since $\dim \ker D_1 < \infty$. Hence there is a decomposition $T_{x_0}\mathcal{B}_{\lambda_0} = F \times \ker D_1$. Now for any $(v, w) \in E$, where $v \in T_{x_0}\mathcal{B}_{\lambda_0}$ and $w \in T_{\lambda_0}\Lambda$, there is a $v_1 \in T_{x_0}\mathcal{B}_{\lambda_0}$ such that $D_1v_1 + D_2w = 0$, where v_1 is uniquely determined up to adding an element of $\ker D_1$. There is a unique choice of v_1 such that $v - v_1 \in F$ under the decomposition $T_{x_0}\mathcal{B}_{\lambda_0} = F \times \ker D_1$. This gives a decomposition of E as $(F \times \{0\}) \times K$, where (v, w) is identified with $((v - v_1, 0), (v_1, w))$. This proves that K splits $T_{x_0}\mathcal{B}$. □

With Lemma 2.30 at hand, we go back to the proof of Theorem 2.29. It follows immediately that $\mathcal{M} := s^{-1}(0)$ is an embedded submanifold of $\mathcal{B}$ because D is surjective by assumption and K splits $T_{x_0}\mathcal{B}$ (cf. Theorem 2.11). Furthermore, $pr|_{\mathcal{M}}$ is Fredholm because both the kernel and cokernel of $D(pr)(x_0)|_K$ are finite dimensional by Lemma 2.30. For part (2), we let $\Lambda_{reg} := \Lambda_{reg}(pr|_{\mathcal{M}})$. Then by Sard-Smale, Λ_{reg} is of Baire's second category. By Lemma 2.30, $\lambda \in \Lambda_{reg}$ iff s_λ is transverse to the zero section (because the cokernel of $D(pr)(x_0)|_K$ is identified via D_2 to the coker D_1). □

Suppose $\lambda_0, \lambda_1 \in \Lambda_{reg}$, and $\gamma : [0, 1] \to \Lambda$ is a smooth path such that $\gamma(i) = \lambda_i$, $i = 0, 1$. We set

$$W_\gamma := \{(x, t) \in \mathcal{M} \times [0, 1] | pr(x) = \gamma(t)\}.$$

It is well-known from the finite dimensional transversality theory that in the case $pr : \mathcal{M} \to \Lambda$ is between finite dimensional spaces, one can perturb γ slightly to make it transverse to pr, so that W_γ is a smooth submanifold with boundary in $\mathcal{M} \times [0, 1]$, providing a "cobordism" between $pr^{-1}(\lambda_0)$ and $pr^{-1}(\lambda_1)$. This is continue to be true in the present infinite dimensional setting.

Theorem 2.31. *Suppose γ_0 is a smooth path connecting $\lambda_0, \lambda_1 \in \Lambda_{reg}$. Then one can slightly perturb γ_0 to a smooth path γ connecting λ_0, λ_1 such that W_γ is a smooth, finite dimensional submanifold with boundary in $\mathcal{M} \times [0, 1]$.*

Exercise 2.32. Prove Theorem 2.31.

Hint: Find an appropriate, finite dimensional space of perturbations P for γ_0 in the following sense: there is $p_0 \in P$ and a smooth map $\Gamma : [0, 1] \times P \to \Lambda$ such that (1) $\Gamma|_{[0,1]\times\{p_0\}} = \gamma_0$, (2) Γ is transverse to $pr : \mathcal{M} \to \Lambda$. Then prove that

$$W_\Gamma := \{(x, t, p) \in \mathcal{M} \times [0, 1] \times P | pr(x) = \Gamma(t, p)\}$$

is a smooth, finite dimensional submanifold with boundary in $\mathcal{M} \times [0,1] \times P$. Finally, apply Sard's theorem to the projection $W_\Gamma \to P$. (cf. [3], section 4.3.)

2.4. **Determinant line bundles and orientations.** Suppose we are given a pair of Banach bundles $\mathcal{E}, \mathcal{F} \to \mathcal{B}$ and a continuous family of Fredholm operators parametrized by $\mathcal{B}$,

$$\mathcal{L} := \{L_x \in \text{Fred }(\mathcal{E}_x, \mathcal{F}_x) | x \in \mathcal{B}\},$$

where for each $x \in \mathcal{B}$, $\mathcal{E}_x, \mathcal{F}_x$ are *real* Banach spaces (i.e., over $\mathbb{R}$). Note that one can regard $\mathcal{L}$ as a continuous section of the associated Banach bundle $L(\mathcal{E}, \mathcal{F}) \to \mathcal{B}$, where $L(\mathcal{E}, \mathcal{F})$ is the bundle whose fiber at $x \in \mathcal{B}$ is $L(\mathcal{E}_x, \mathcal{F}_x)$, the Banach space of bounded linear operators from $\mathcal{E}_x$ to $\mathcal{F}_x$. The purpose of this section is to construct a *real* topological line bundle over $\mathcal{B}$, called *the determinant line bundle* of $\mathcal{L}$, which will be denoted by $\det \mathcal{L}$.

Let V be a real vector space of dimension n. We denote by $\Lambda^{max} V = \Lambda^n V$ the n-th exterior power of V, which is a 1-dimensional real vector space. If V^* denotes the dual space of V, then one has a canonical isomorphism $(\Lambda^{max} V)^* = \Lambda^{max} V^*$. With this understood, for each $x \in \mathcal{B}$, since L_x is Fredholm, both $\ker L_x$ and coker L_x are finite dimensional real vector spaces. The determinant line bundle $\det \mathcal{L}$ is defined by setting the fiber at x to be

$$(\det \mathcal{L})_x := \Lambda^{max}(\ker L_x) \otimes (\Lambda^{max}(\text{coker } L_x))^*.$$

Theorem 2.33. *(cf. Donaldson-Kronheimer* [3]*) Let $\mathcal{L}$ be a continuous family of Fredholm operators parametrized by $\mathcal{B}$. Then* $\det \mathcal{L}$ *forms a topological line bundle over $\mathcal{B}$.*

Proof. Let $x_0 \in \mathcal{B}$ be any point. There exists a neighborhood U of x_0 such that $\mathcal{E}|_U, \mathcal{F}|_U$ are trivial. After fixing a trivialization $U \times E$, $U \times F$ of $\mathcal{E}|_U, \mathcal{F}|_U$ respectively, we may regard L_x is a continuous family of Fredholm operators from fixed Banach spaces E to F, $x \in U$. We shall first consider the simplified situation where L_{x_0} is surjective (for all x_0). In this case, since $\ker L_{x_0}$ is finite dimensional, there exists a closed subspace E_1 of E such that E can be written as $\ker L_{x_0} \times E_1$. We let $P : E \to E_1$ be the projection, and $T : F \to E_1$ be the bounded linear operator such that $L_{x_0} \circ T = Id$, $T \circ L_{x_0} = P$. Then we have, for any $u \in \ker L_x$,

$$0 = T \circ L_x u = T \circ L_{x_0} u + T(L_x - L_{x_0})u = Pu + Q(x)u,$$

where $Q(x) := T(L_x - L_{x_0}) : E \to E_1 \subset E$ such that $||Q(x)||$ can be made arbitrarily small when x is sufficiently close to x_0. This implies that the projection onto $\ker L_{x_0}$, $Id - P = Id + Q$, when restricted to $\ker L_x$, is an isomorphism when x is sufficiently close to x_0. Let us denote its inverse by $f_x : \ker L_{x_0} \to \ker L_x$. Then for any given basis $v = (v_1, v_2, \cdots, v_n)$ (here $n = \dim \ker L_{x_0}$) of $\ker L_{x_0}$, there is an element $f_x(v) := f_x(v_1) \wedge \cdots \wedge f_x(v_n) \in (\det \mathcal{L})_x$. If we call the fixed trivializations of $\mathcal{E}, \mathcal{F}$ over U collectively by ϕ, then this construction gives us a bijective map which is linear over each fiber:

$$\Psi_{x_0,v,\phi} : \det \mathcal{L}|_U \to U \times \mathbb{R}, \;\; t f_x(v) \mapsto (x, t), \forall x \in U.$$

By the "Vector Bundle Construction Lemma", our claim that $\det \mathcal{L}$ is a line bundle over $\mathcal{B}$ follows immediately if any transition maps $\Psi_{x_0,v,\phi} \circ \Psi^{-1}_{y_0,w,\psi}$ are continuous, i.e.,

$\Psi_{x_0,v,\phi} \circ \Psi^{-1}_{y_0,w,\psi} : (x,t) \mapsto (x, \rho(x)t)$ for some continuous function $\rho(x)$, which can be easily verified.

In the general case where L_{x_0} is not necessarily surjective for every $x_0 \in \mathcal{B}$, we employ the idea of "stabilization". We need the following lemma first.

Lemma 2.34. *(1) Let W be a finite dimensional vector space and $V \subset W$ be a subspace. Then there is a canonical isomorphism $\Lambda^{max} W = \Lambda^{max} V \otimes \Lambda^{max}(W/V)$.*

(2) Let $0 \rightarrow E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \xrightarrow{k} E_4 \rightarrow 0$ be an exact sequence of finite dimensional vector spaces. Then there is an canonical isomorphism (determined by i,j,k)

$$\Lambda^{max} E_1 \otimes (\Lambda^{max} E_4)^* = \Lambda^{max} E_2 \otimes (\Lambda^{max} E_3)^*.$$

Proof. (1) Let $\dim W = n$, $\dim V = m$. Let $V_1 \subset W$ be any subspace such that $W = V \oplus V_1$. Then there is an embedding $V^{\otimes m} \otimes V_1^{\otimes(n-m)} \hookrightarrow W^{\otimes n}$, which induces an isomorphism $\Lambda^m V \otimes \Lambda^{(n-m)} V_1 \rightarrow \Lambda^n W$. Now by identifying W/V with V_1, we obtain an isomorphism $\Lambda^m V \otimes \Lambda^{(n-m)}(W/V) \rightarrow \Lambda^n W$, which is canonical because it does not depend on the choice of V_1.

(2) Since $i : E_1 \rightarrow E_2$ is injective, we can regard E_1 as a subspace of E_2 via i. Likewise, since $k : E_3 \rightarrow E_4$ is surjective, we can regard E_4^* as a subspace of E_3^* via k^*. By part (1) of the lemma, we obtain isomorphisms

$$\Lambda^{max} E_2 = \Lambda^{max} E_1 \otimes \Lambda^{max}(E_2/E_1), \;\; \Lambda^{max} E_3^* = \Lambda^{max} E_4^* \otimes \Lambda^{max}(E_3^*/E_4^*).$$

Now E_3^*/E_4^* is naturally identified with the dual space of the image of $j : E_2 \rightarrow E_3$, which via j, is identified to the dual space of E_2/E_1. Hence

$$\begin{aligned} \Lambda^{max} E_2 \otimes \Lambda^{max} E_3^* &= \Lambda^{max} E_1 \otimes \Lambda^{max}(E_2/E_1) \otimes \Lambda^{max} E_4^* \otimes \Lambda^{max}(E_2/E_1)^* \\ &= \Lambda^{max} E_1 \otimes \Lambda^{max} E_4^* \otimes \Lambda^{max}(E_2/E_1) \otimes \Lambda^{max}(E_2/E_1)^* \\ &= \Lambda^{max} E_1 \otimes \Lambda^{max} E_4^* \otimes \mathbb{R} \\ &= \Lambda^{max} E_1 \otimes \Lambda^{max} E_4^*. \end{aligned}$$

□

Now back to the proof of Theorem 2.33. For any $x_0 \in \mathcal{B}$, since coker L_{x_0} is finite dimensional, there exists a linear map $\xi : \mathbb{R}^N \rightarrow F$ for some N such that $L_{x_0} \oplus \xi : E \oplus \mathbb{R}^N \rightarrow F$ is surjective. Since surjectivity is an open condition, there exists a neighborhood U of x_0 such that $L_x \oplus \xi : E \oplus \mathbb{R}^N \rightarrow F$ is surjective for all $x \in U$. Now consider the exact sequence

$$0 \rightarrow \ker L_x \xrightarrow{i} \ker(L_x \oplus \xi) \xrightarrow{j} \mathbb{R}^N \xrightarrow{k} \text{coker } L_x \rightarrow 0,$$

where i is the inclusion induced by $E \hookrightarrow E \oplus \mathbb{R}^N$, j is the inclusion $\ker(L_x \oplus \xi) \subset E \oplus \mathbb{R}^N$ followed by the projection onto $\mathbb{R}^N$, and k is $\xi : \mathbb{R}^N \rightarrow F$ followed by the projection from F onto coker L_x. By Lemma 2.34(2), there is an canonical isomorphism $I_{x,\xi} : \Lambda^{max} \ker(L_x \oplus \xi) \otimes (\Lambda^{max} \mathbb{R}^N)^* \rightarrow \Lambda^{max}(\ker L_x) \otimes (\Lambda^{max}(\text{coker } L_x))^*$. Apply the previous construction to the family of Fredholm operators $L_x \oplus \xi : E \oplus \mathbb{R}^N \rightarrow F$, we obtain a trivialization

$$\Psi_{x_0,v,\phi,\xi} : \det \mathcal{L}|_U \rightarrow U \times \mathbb{R}, \;\; t I_{x,\xi}(f_x(v)) \mapsto (x,t), \forall x \in U.$$

The claim that $\det \mathcal{L}$ is a topological line bundle over $\mathcal{B}$ follows by verifying that the transition maps $\Psi_{x_0,v,\phi,\xi} \circ \Psi^{-1}_{y_0,w,\psi,\eta}$ are continuous. $\square$

Remark 2.35. (1) If $\mathcal{L}$ is a smooth family of Fredholm operators parametrized by $\mathcal{B}$, then $\det \mathcal{L}$ is a smooth line bundle over $\mathcal{B}$.

(2) If the fibers of $\mathcal{E}, \mathcal{F}$ are the underlying real Banach spaces of a complex Banach space, and each L_x is complex linear, then the determinant line bundle $\det \mathcal{L} \to \mathcal{B}$ must be trivial, and moreover, there is a canonical trivialization of $\det \mathcal{L}$ coming from the complex structures, cf. [10], p. 500.

(3) Let $s : \mathcal{B} \to \mathcal{E}$ be a family of Fredholm sections, $s = \{s_\lambda : \mathcal{B}_\lambda \to \mathcal{E}_\lambda\}$, parametrized by $\lambda \in \Lambda$. Then for any $x \in \mathcal{M} := s^{-1}(0)$, suppose $x \in \mathcal{B}_\lambda$ for some λ, there is a Fredholm operator $Ds(x) : T_x\mathcal{B}_\lambda \to \mathcal{E}_\lambda|_x$, which is given by $Df(x)$ if s_λ is locally given by the graph of f. Note that the well-definedness of $Ds(x)$ (i.e., independent of the choice of f) requires the fact that $x \in s^{-1}(0)$. We thus have a smooth family of Fredholm operators $Ds = \{Ds(x)\}$ parametrized by $x \in \mathcal{M} = s^{-1}(0)$, and correspondingly the determinant line bundle $\det(Ds) \to \mathcal{M}$. Now assume $\det(Ds) \to \mathcal{M}$ is trivial and s is transverse to the zero section. Then by Theorem 2.29, $\mathcal{M}$ is an embedded submanifold of $\mathcal{B}$ and for any $\lambda \in \Lambda_{reg}$ (a subset of Λ of Baire's second category), $\mathcal{M}_\lambda := s_\lambda^{-1}(0)$ is a finite dimensional smooth manifold. The key fact is that for any $x \in \mathcal{M}_\lambda$, $\det(Ds)|_x = \Lambda^{max} T_x\mathcal{M}_\lambda$, so that the triviality of $\det(Ds) \to \mathcal{M}$ implies that $\mathcal{M}_\lambda$ is orientable for any $\lambda \in \Lambda_{reg}$. Moreover, a fixed trivialization of $\det(Ds) \to \mathcal{M}$ gives rise to a coherent orientation to each $\mathcal{M}_\lambda$, $\lambda \in \Lambda_{reg}$, in the following sense: for any $\lambda_0, \lambda_1 \in \Lambda_{reg}$, suppose $\gamma : [0,1] \to \Lambda$ is a smooth path connecting λ_0, λ_1 such that $W_\gamma := \{(x,t) \in \mathcal{M} \times [0,1] | pr(x) = \gamma(t)\}$ is a smooth embedded submanifold of boundary in $\mathcal{M} \times [0,1]$ (cf. Theorem 2.31). Then there is an orientation on W_γ such that the induced orientation on the boundary $\partial W_\gamma = \mathcal{M}_{\lambda_0} \sqcup \mathcal{M}_{\lambda_1}$ coincides with the orientation obtained from the determinant line bundle $\det(Ds) \to \mathcal{M}$.

(4) Let $s : \mathcal{B} \to \mathcal{E}$ be a family of Fredholm sections, $s = \{s_\lambda : \mathcal{B}_\lambda \to \mathcal{E}_\lambda\}$, parametrized by $\lambda \in \Lambda$. In the situations which we will encounter, $Ds(x) : T_x\mathcal{B}_\lambda \to \mathcal{E}_\lambda|_x$ can be defined for any $x \in \mathcal{B}$ and is Fredholm. This gives rise to a determinant line bundle $\det(Ds) \to \mathcal{B}$ which restricts to $\det(Ds) \to \mathcal{M}$. The point of introducing this line bundle on the larger space $\mathcal{B}$ is that, in general, one knows nothing about the topology of $\mathcal{M}$ but often has a good grasp of the topology of $\mathcal{B}$, hence it is often much easier to show that $\det(Ds) \to \mathcal{B}$ is trivial. The triviality of $\det(Ds) \to \mathcal{B}$ then implies the triviality of $\det(Ds) \to \mathcal{M}$, which implies that each $\mathcal{M}_\lambda$, $\lambda \in \Lambda_{reg}$, can be coherently oriented.

Exercise 2.36. Work out the details of the claims in Remark 2.35(1), (2), (3).

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