Midterm Exam
Stat 697U, Spring 2019

Instructor: Brian Van Koten

Due Friday, March 29, in my mailbox or by email before 3:00pm

Please do not discuss the problems on this exam with anyone. You may consult any of the recommended texts, but please do not search the web for solutions.

Problem 1 (3+2 points). *Let E be a countable set, and let X_0 be a random variable with values in E. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed uniform random variables on the interval $[0,1]$. Suppose that $G : E \times [0,1] \to E$, and define a discrete time process inductively by*

$$X_{n+1} = G(X_n, U_{n+1}).$$

1. *Show that X_n is a homogeneous discrete time Markov chain, and write down its transition matrix in terms of G.*
2. *Can every homogeneous discrete time Markov chain be written in this fashion for some function G?*

Problem 2 (5 points). *Let $\{D_n\}_{n=1}^{\infty}$ be the outcomes of independent rolls of a fair die. (That is, the D_n's are i.i.d. uniform on $\{1,\dots,6\}$). Let*

$$S_n = \sum_{k=1}^{n} D_k$$

be the sum of the first n rolls. Compute

$$\lim_{n\to\infty} \mathbb{P}[S_n \textit{ is a multiple of } 13].$$

Hint: *Is the remainder R_n of S_n on division by 13 a Markov chain?*

Problem 3 (4+1 points). *Let $\{X_n\}_{n=1}^{\infty}$ be a Markov chain on a finite state space E with transition matrix P. Fix $k \in E$, and let $H^k \in [0,\infty]^E$ be the first hitting time of k,*

$$H^k := \inf\{n \geq 0 : X_n = k\}.$$

(Note that this is the first hitting *time, not the first* passage *time.) Fix $j \in E$ with $j \neq k$, and define*

$$g_i^{jk} = \mathbb{P}_i[H^k < H^j].$$

1. *Find a system of linear equations analogous to [Norris, Theorem 1.3.2] whose solution is the vector* $g^{jk} \in [0,1]^E$ *whose i'th entry is* g_i^{jk}.

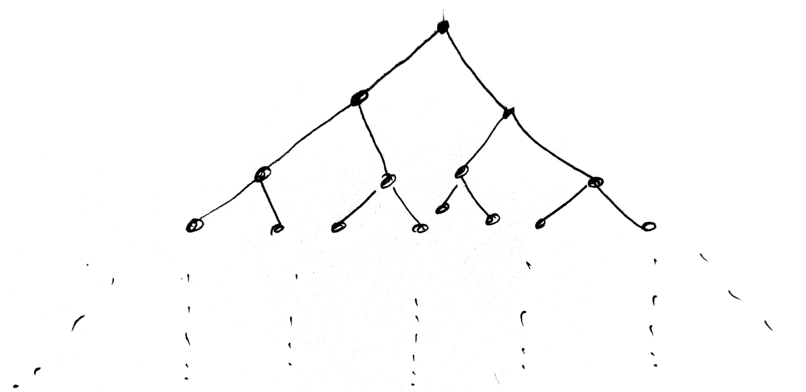
   Hint: *You may use [Norris, Theorem 1.3.2] in the solution of this problem, but you will have to apply it to a modified version of the original Markov chain* X_n. *Otherwise, you can use a first-step analysis as in the proof of [Norris, Theorem 1.3.2].*

2. *Write the system of equations in the matrix form*

$$Mg^{jk} = b$$

   for some matrix M *and vector* b.

Problem 4 (3+2 points). *Consider the infinite binary tree in the figure below:*



This tree has a single vertex, called the root, with degree two. All other vertices have degree exactly three, and there are no cycles (closed loops) in the graph.

1. *Suppose that we define a random walk on the graph by starting at the root and at each step following one edge chosen uniformly at random. (See Homework 5 for a more precise description of the random walk on a graph.) Show that the random walk on the infinite binary tree is transient.*

2. *Suppose that instead of choosing an edge uniformly at random, we choose the edge that leads in the direction of the root with probability* $1/2$ *and the edges that lead away from the root with probability* $1/4$ *each. (At the root, we simply choose one of the edges leading away, each with probability* $1/2$.*) Is this walk transient?*