Homework 9
Stat 697U, Spring 2019

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Due Wednesday, April 17

Problem 1 (2+3 points). *Let p_λ be the density of the exponential distribution with parameter λ. Recall that the mean of this distribution is λ^{-1} and the variance is λ^{-2}. Suppose we wish to compute the integral*

$$I = \int_0^\infty xe^{-x}\,dx.$$

1. *How would you estimate I using importance sampling based on i.i.d. samples drawn from p_λ?*

2. *As a function of λ, what is the variance of the importance sampling estimate of I? For what value of λ is the variance the smallest?*

   Hints: You will not have to evaluate any integrals if you use the information above about the mean and variance of the exponential distribution. You should find that the variance is infinite for some values of λ!

Sometimes one wants to do importance sampling in cases when the normalizing constants of the target and instrumental distributions are unkown. Let π (resp. ν) be the target (resp. instrumental) distribution, and assume that the density of π (resp. ν) is $Z^{-1}p(x)$ (resp. $W^{-1}q(x)$). Suppose that the objective is to compute $\mathbb{E}_\pi[f]$ for some function f. In this case, one can use the *self-normalized importance sampling estimator*

$$\tilde{f}_N^q = \frac{\sum_{i=1}^N f(Y_i)\frac{p(Y_i)}{q(Y_i)}}{\sum_{i=1}^N \frac{p(Y_i)}{q(Y_i)}},$$

where $Y_1, \ldots, Y_n$ are i.i.d. random variables with distribution ν.

Problem 2 (2 points). *Use the strong law of large numbers to show that*

$$\mathbb{P}\left[\lim_{N\to\infty} \tilde{f}_N^q = \mathbb{E}_\pi[f]\right] = 1.$$

Hint: *You may use the following theorem without proof: If $V_n \to V$ with probability one and $W_n \to C > 0$ with probability one, then $V_n/W_n \to V/C$ with probability one.*

Consider the model of a polymer as a self-avoiding walk introduced in Wednesday's lecture: The set of self-avoiding walks is

$$\Omega_M := \{x \in (\mathbb{Z}^2)^{M+1}; x_0 = (0,0), x_k \neq x_i \text{ for any } i < k, \|x_k - x_{-1}\| = 1 \text{ for } k = 1, \ldots, M\}.$$

We let $p_M(x)$ the probability mass function of the uniform distribution on Ω_M. One can generate self-avoiding walks by the following algorithm:

1. Set $x_0 = (0,0)$.

2. Given x_k, let x_{k+1} be drawn uniformly at random from among the *unoccupied* nearest neighbors of x_k. That is, let x_{k+1} be drawn uniformly at random from the set $\{z \in \mathbb{Z}^2; \|z - x_k\| = 1,\ z \neq x_i \text{ for any } i = 1, \ldots, k\}$. If there are no unoccupied nearest neighbors, start over again at $(0,0)$. Otherwise, iterate until a self-avoiding walk of length M has been generated.

Problem 3 (2 + 4 points)**.** *The output of the above algorithm for generating self-avoiding walks is a random variable on Ω_M. Let $q_M : \Omega_M \to [0,1]$ be the probability mass function of this random variable.*

1. *Show that $p_4 \neq q_4$ by giving an example of two self-avoiding walks of length 4 with different probablities under q_4.*

2. *Derive an importance sampling algorithm for computing*

$$\mathbb{E}_{p_M}[\|x_M - x_0\|^2]$$

using q_M as the instrumental distribution.

Hint: *You will have to use the self-normalized estimator.*