Homework 9.2
Stat 605, Fall 2018

Instructor: Brian Van Koten

Due Thursday, November 15

Let X be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$. Next week we will see that if $\mathbb{E}[|X|] < \infty$, then a conditional expectation $\mathbb{E}[X|\mathscr{G}]$ exists for any σ-algebra $\mathscr{G} \subset \mathscr{F}$. The conditional expectation is unique in the sense that if Y and Y' are both versions of $\mathbb{E}[X|\mathscr{G}]$, then $Y = Y'$ almost surely. Moreover, if Y is a version of $\mathbb{E}[X|\mathscr{G}]$ and $Y = Y'$ almost surely, then Y' is a version of $\mathbb{E}[X|\mathscr{G}]$. **Therefore, any equalities or inequalities involving conditional expectations are to be understood as holding almost surely, even when this is not explicitly stated.**

Problem 1 (2 points)**.** *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and suppose that $\Omega_1, \ldots, \Omega_n$ be a partition of Ω, i.e. the sets $\Omega_1, \ldots, \Omega_n$ are disjoint and $\cup_{i=1}^n \Omega_i = \Omega$. Let $\mathscr{G} = \sigma(\Omega_1, \ldots, \Omega_n)$ be the σ-algebra generated by these sets. Prove that for any random variable X,*

$$\mathbb{E}[X|\mathscr{G}] = \frac{\mathbb{E}[X\mathbf{1}_{\Omega_i}]}{\mathbb{P}[\Omega_i]}(\omega) \text{ for } \omega \in \Omega_i.$$

Problem 2 (3 points)**.** *Let Y be an exponential random variable with parameter λ, i.e. $\mathbb{P}[Y \geq \alpha] = \exp(-\lambda\alpha)$.*

1. *Prove that Y is* memoryless*, that is for any $T, t > 0$,*

   $$\mathbb{P}[Y \geq T + t | Y \geq T] = \mathbb{P}[Y \geq t].$$

   (To understand why we use the word memoryless here, imagine that Y is the lifetime of a lightbulb, for example.)

2. *Find a convenient formula expressing $\mathbb{E}[Y|\min\{Y, T\}]$ in terms of $\min\{Y, T\}$.*

Problem 3 (3 points)**.** *Let X and Y be independent random variables on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and assume that X is integrable. Show that $\mathbb{E}[X|Y] = \mathbb{E}[X]$.*

Problem 4 (3 points)**.** *Let X be an integrable random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra. Show that $\mathbb{E}[X|\mathscr{G}]$ is integrable.*

Problem 5 (3 points)**.** *Let $X_1, \ldots, X_n$ be i.i.d. integrable random variables on $(\Omega, \mathscr{F}, \mathbb{P})$. Show that $\mathbb{E}[X_j|\sum_{i=1}^n X_i] = n^{-1}\sum_{i=1}^n X_i$.*