Homework 8
Stat 697U, Spring 2019

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Due Wednesday, April 10

Recall Theorem 2.4.4 from Norris's book on the superposition of independent Poisson processes. The following problem provides an inverse operation to superposition. This inverse operation is called *thinning*.

Problem 1 (3+3 points)**.** *Let* $\{X_t\}_{t\in[0,\infty)}$ *be a Poisson process with rate* λ*. Let* $\{B_n\}_{n=1}^{\infty}$ *be i.i.d. Bernoulli with success probability* p*, and assume that* $\{B_n\}_{n=1}^{\infty}$ *is independent of* $\{X_t\}_{t\in[0,\infty)}$*. Define thinned processes*

$$X_t^1 = \sum_{n=1}^{X_t} B_n, \text{ and}$$

$$X_t^0 = X_t - X_t^1 = \sum_{n=1}^{X_t}(1 - B_n).$$

A more concrete definition of the thinned processes is as follows: Suppose that the original process X_t *counts the number of events, say calls placed at a telephone exchange, up to time* t*. Suppose that these events can be classified into two types 1 and 0, say calls to neighborhoods 1 and 0. If the classifications of the events are independent of the each other and the process of event times* X_t*, and each event is classified as type 1 with probability* p*, then* X_t^1 *is the number of type 1 events up to time* t *and* X_t^0 *is the number of type 0 events up to time* t*.*

1. *Show that* X_t^1 *is a Poisson process with rate* λp *and that* X_t^0 *is a Poisson process with rate* $\lambda(1-p)$*.*

   Hint: *Try using the infinitesimal definition of the Poisson process.*

2. *Show that* X_t^1 *and* X_s^0 *are independent for each* $t, s \in [0,\infty)$*.*

   Note: *In fact, the two thinned processes are independent. By definition, that means the every event determined by one process is independent of every event determined by the other. There are measure-theoretic subtleties involved here, so you only have to prove the simple statement above.*

Here is a problem on Poisson processes from Norris's book.

Problem 2 (2+2+2 points)**.** *A pedestrian wishes to cross a single lane of fast-moving traffic. Suppose that the number of vehicles that hae passed by time ti sia Poisson process of rate* λ*, and suppose it takes time* a *to walk across the lane.*

1. *Assuming that the pedestrian can foesee correctly the times at which vehicles will pass by, how long on average does it take to cross over safely?*
2. *How long on average does it take to cross two similar lanes when one must walk straight across, assuming that the pedestrian will not cross if, at any time while crossing, a car would pass in either direction?*
3. *How long does it take to cross when an island in the road makes it safe to stop half-way?*

Problem 3 (1 point)**.** *Let* $\alpha, \beta > 0$*, and define the generator matrix*

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

Find a formula expressing the unique invariant distribution of Q *in term of* α *and* β*.*

Here is another problem from Norris's book. This problem relates to the study of queues, which has important applications in computer science and engineering. Instead of going through the theory of queues in this class, I'm going to do Monte Carlo methods. But let me know if you like queues, and we can discuss.

Problem 4 (3+3+3+3+3+1 points)**.** *Customers arrive at a certain queue in a Poisson process of rate* λ*. The single server has two states* A *and* B*, state* A *signifying that he is in attendance and state* B *that he is having a coffee break. Independently of how many customers are in the queue, he fluctuates between these states as a Markov chain* $\{Y_t\}$ *on* A, B *with the generator*

$$\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

The total service time for any customer is exponentially distributed with parameter μ *and is independent of the chain* $\{Y_t\}$ *and of the service times of other customers.*

1. *Show that the system is a continuous time Markov chain* $\{X_t\}$ *with state space*

   $$\{A_0, A_1, A_2, \dots\} \cup \{B_0, B_1, B_2, \dots\},$$

   where A_n *signifies that there are* n *people in the queue and the server is in attendance and* B_n *signifies that there are* n *people in the queue and the server is having a coffee break. Write down the generator matrix of this Markov chain.*

2. *Show that* $\mathbb{P}[\text{hit } A_0 | X_0 = A_k] = \theta^k$ *for some* $\theta \in (0, 1]$.

3. *Show that* $\mathbb{P}[\text{hit } A_k | X_0 = B_{k+1}] = \frac{\beta\theta}{\lambda+\beta-\lambda\theta}$.

   Hint: *Let* E_m *be the event that exactly* m *customers arrive while the server is on his first coffee break. One approach is to begin by writing*

$$\mathbb{P}_{B_{k+1}}[\text{hit } A_k] = \sum_{m=0}^{\infty} \mathbb{P}_{B_{k+1}}[\text{hit } A_k, E_m].$$

   This is a geometric series, and it is easy to compute the limit.

4. *Show that* $(\theta - 1) f(\theta) = 0$, *where*

$$f(\theta) = \lambda^2\theta^2 - \lambda(\lambda + \mu + \alpha + \beta)\theta + (\lambda + \beta)\mu.$$

   Hint: *Do a first-step analysis for the jump chain. Use part 3.*

5. *Prove that* $\{X_t\}$ *is transient if* $\frac{\mu\beta}{\alpha+\beta} < \lambda$.

6. *Give an interpretation of part 5, explaining why the result is intuitively obvious.*

   Hint: *There are many good answers. You might think about problem 3, the continuous time Markov chain with two states. Given the formula proved in problem 3, what do you imagine is the effective rate at which customers are served? What is the average rate at which customers arrive?*