Homework 8.2
Stat 605, Fall 2018

Instructor: Brian Van Koten

Due Thursday, November 8

Problem 1 (3 points)**.** *Let X be multivariate normal. Show that if two coordinate components X_i and X_j of X are uncorrelated, then X_i and X_j are independent.*

Problem 2 (3 points)**.** *Let $Q \in \mathbb{R}^{d\times d}$ be a nonnegative semidefinite, symmetric matrix, and let $\mu \in \mathbb{R}^d$. Let $D \in \mathbb{R}^{m\times d}$ be an arbitrary matrix. Show that if $\mathcal{L}(Z) = N(\mu, Q)$, then $\mathcal{L}(DZ) = N(D\mu, DQD^{\mathrm{t}})$.*

Recall Slutsky's theorem from the last homework:

Theorem 1. *Suppose that $\{X_n\}_{n=1}^{\infty}$, X are $\mathbb{R}^d$-valued random variables and that $\{Y_n\}_{n=1}^{\infty}$, Y are $\mathbb{R}^m$-valued random variables. If $X_n \xrightarrow{\mathscr{D}} X$ and $Y_n \xrightarrow{\mathscr{D}} c$ where $c \in \mathbb{R}^m$ is a constant, then $(X_n, Y_n) \xrightarrow{\mathscr{D}} (X, c)$.*

Slutsky's theorem is of fundamental importance in the study of asymptotic properties of estimators in statistics. To see how it is typically applied, prove the following result, which is sometimes called the *delta method*:

Problem 3 (5 points)**.** *Let $\{Y_n\}_{n=1}^{\infty}$ be $\mathbb{R}$-valued random variables so that something like the CLT holds:*

$$\sqrt{n}(Y_n - \mu) \xrightarrow{\mathscr{D}} W,$$

for some constant μ and random variable W. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable at μ. Show that we have

$$\sqrt{n}(g(Y_n) - g(\mu)) \xrightarrow{\mathscr{D}} g'(\mu)W.$$

Hint: *This problem is intended to test not only the use of Slutsky's theorem, but also the continuous mapping theorem and Taylor expansions similar to those used to prove the CLT: Begin by justifying the expansion*

$$g(Y_n) - g(\mu) = g'(\mu)(Y_n - \mu) + h(Y_n - \mu)(Y_n - \mu),$$

where $\lim_{u\to 0} h(u) = 0$.

If you prefer, you can prove the multivariate version of the delta method: Both the proof and the statement are essentially identical with the one-dimensional case. Combining the multivariate delta method with Problem 2 yields the following interesting result: Let $\{Z_n\}_{n=1}^{\infty}$ be $\mathbb{R}^d$-valued random variables such that $\sqrt{n}(Z_n - \mu) \to N(0, Q)$. Let $g : \mathbb{R}^d \to \mathbb{R}^m$ be differentiable at μ. Then we have

$$\sqrt{n}(g(Z_n) - g(\mu)) \to N(0, g'(\mu)Qg'(\mu)^{\mathrm{t}}).$$