Homework 8.1
Stat 605, Fall 2018

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Due Thursday, November 8

Recall that I did not quite give the entire proof of our theorem on derivatives of characteristic functions. The following problem will fill in the gap:

Problem 1 (4 points). *Let X be a random variable taking values in $\mathbb{R}^d$, and assume that $\mathbb{E}[|X|^m] < \infty$. Show that the function $g : \mathbb{R}^d \to \mathbb{C}$ by*

$$g(u) = i^m \mathbb{E}[X_{j_1} \dots X_{j_m} \exp(i\langle u, X\rangle)]$$

is continuous. Hint: *Such a function g is continuous at a point $u \in \mathbb{R}^d$ if and only if for every sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ with $u_n \to u$, $g(u_n) \to g(u)$. Use this definition of continuity together with the dominated convergence theorem.*

Problem 2 (3 points). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables taking values in $\mathbb{R}^d$. Suppose that* $\sup_n \mathbb{E}[|X_n|] < \infty$. *Show that $\{\mathcal{L}(X_n)\}_{n=1}^{\infty}$ is tight.*

The Cauchy distribution is the probability measure on $\mathbb{R}$ with the density

$$f(x) = \frac{1}{\pi(1 + x^2)}.$$

Observe that the Cauchy distribution does not have finite first moment, so the law of large numbers does not apply. In fact, as you will show in the following problem, it fails spectacularly!

Problem 3 (2+2 points).

1. *Find a formula for the characteristic function of the Cauchy distribution.*
2. *Show that if $X_1, \dots, X_N$ are i.i.d. Cauchy, then $\frac{1}{N}\sum_{i=1}^{N} X_i$ has the same distribution as X_1!*