Homework 7
Stat 697U, Spring 2019

Instructor: Brian Van Koten

Due Wednesday, March 27

Here is an application of the elementary properties of the exponential distribution proved on the last homework.

Problem 1 (3+1 point). *1. A post office has two clerks. Service times for clerk i are independent* Exponential(λ_i) *for $i = 1, 2$. When you arrive at the post office, both clerks are busy, but no one else is waiting. You will enter service when either clerk becomes free. Find $\mathbb{E}[T]$, where T is the amount of time you spend in the post office.*

2. Suppose that the service times are not exponential, but instead they follow some other distribution with known parameters. Do you still have enough information to solve the problem?

Here is part of a problem from Norris's book:

Problem 2 (3 points). *Two fleas are bound together to take part in a nine-legged race on the vertices A, B, and C of a triangle. Flea 1 hops at random times in the clockwise direction; each hop takes the pair from one vertex to the next and the times between successive hops of Flea 1 are independent random variables, each with exponential distribution, mean $1/\lambda$. Flea 2 behaves similarly, but hops in the counterclockwise direction, the times between his hops having mean $1/\mu$. Show that the location X_t of the fleas at time t is a continuous time Markov chain on the triangle and find its generator matrix Q.*

The following problem gives a general solution to the forwards equation, which is sometimes useful.

Problem 3 (3+2+2 points). *1. For any matrix $A \in \mathbb{R}^{n \times n}$, define the matrix exponential*

$$\exp(A) = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

(Here, I denotes the identity matrix.) Show that for any $A \in \mathbb{R}^{n \times n}$,

$$P(t) = \exp(At)$$

solves the differential equation

$$P'(t) = P(t)A.$$

Note: *If you like, you may simply assume that it is permissible to interchange the derivative with the summation defining the matrix exponential. However, if you know some analysis, I suggest that you try to do it rigorously.*

2. *Whenever you introduce the matrix exponential, you are required to give the following problem: Show that if X and Y are matrices with $XY = YX$, then $\exp(X+Y) = \exp(X)\exp(Y)$.*

3. *More importantly, you are also required to give this problem: Find a pair of matrices X and Y so that $\exp(X+Y) \neq \exp(X)\exp(Y)$.*

4. *Suppose that A is diagonalizable, so*

$$A = V\Lambda V^{-1},$$

where Λ is a diagonal matrix. Show that $\exp(A) = VMV^{-1}$, where M is a diagonal matrix so that $M_{ii} = \exp(\Lambda_{ii})$.

Problem 4 (3 points)**.** *Let $Q \in \mathbb{R}^{E\times E}$ be a finite matrix. Assume that Q has the properties of a generator matrix. That is, let*

$$Q_{ii} < 0 \text{ and } Q_{ij} \geq 0 \text{ for all } i, j \in E,$$

and let

$$\sum_{j\in E} Q_{ij} = 0 \text{ for all } i \in E.$$

Show that $\exp(tQ)$ is stochastic for all $t \geq 0$.

Note: *Please do this directly, without referring to Markov chains or the forwards equation.*