Homework 7.2
Stat 605, Fall 2018

Instructor: Brian Van Koten

Due Tuesday, October 29

The following problem makes a connection between rare events and the exponential distribution. Recall that X is exponentially distributed with parameter λ if

$$F_X(y) = 1 - \exp(-\lambda y).$$

Problem 1 (5 points)**.** *Let $\{X_i^p\}_{i=1}^{\infty}$ be independent Bernoulli trials with success probability $p > 0$. Let N_p be the number of trials required before the first success, i.e.*

$$N_p = \min\{i : X_i^p = 1\}.$$

(Recall from previous homework that N_p has the geometric distribution.) Prove that $pN_p \xrightarrow{\mathscr{D}} L$ as $p \to 0$, where L has the exponential distribution with parameter $\lambda = 1$. Hint: *Use the characterization of weak convergence in terms of convergence of distribution functions.*

A function $g : \mathbb{R}^d \to \mathbb{R}^q$ is said to be Lipschitz continuous if there exists a constant $L > 0$ so that for every $x, y \in \mathbb{R}^d$,

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

For example, if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $|f'(x)| \leq C$ for all $x \in \mathbb{R}$, then by the mean value theorem,

$$|f(x) - f(y)| = |f'(a)(x - y)| = |f'(a)||x - y|$$

for some $a \in [x, y]$, so f is Lipschitz with constant C.

It turns out that to verify convergence in distribution, it will suffice to prove convergence of expectations for Lipschitz continuous functions:

Theorem 1 (Jacod and Protter, Theorem 18.7)**.** *We have $X_n \xrightarrow{\mathscr{D}} X$ if and only if for every bounded Lipschitz continuous function g, $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$.*

In fact, it suffices to prove convergence of expectations for even smaller classes of functions. For example, one can use the class C_c^{∞} consisting of infinitely differentiable functions taking nonzero values only inside a bounded set.

Problem 2 (5 points)**.** *Prove Slutsky's theorem: Suppose that* $X_n \xrightarrow{\mathscr{D}} X$ *and* $Y_n \xrightarrow{\mathscr{D}} c$ *where* $c \in \mathbb{R}^d$ *is a constant. Then we have* $(X_n, Y_n) \xrightarrow{\mathscr{D}} (X, c)$. Hint: *Use the characterization of weak convergence based on Lipschitz continuous functions, and use that* $Y_n \xrightarrow{\mathscr{D}} c$ *implies* $Y_n \xrightarrow{\mathrm{P}} c$.