Homework 6
Stat 697U, Spring 2019

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Due Wednesday, March 6

Here is a problem introducing branching processes, which are an important class of Markov chain. The solution uses generating functions, which are the subject of Section 1.5.1 in Brémaud's book. You might consider whether it would be feasible to solve the problem using the first-step analysis techniques of Section 1.5 in Norris's book.

Problem 1 (2+3+2+2 points). *In 1874, preoccupied by the extinction of aristocratic family names, Francis Galton and Henry William Watson proposed a Markov chain of the form*

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i},$$

where the $Z_{n,i}$'s are i.i.d. random variables taking values in $\{0, 1, 2, \dots\}$. (The case where $X_{n-1} = 0$ may be confusing: We set $X_n = 0$ when $X_{n-1} = 0$.) In Galton and Watson's model, X_n represents the number of men with a certain family name in the n'th generation, and $Z_{n,i}$ represents the number of sons of the i'th man in the $(n-1)$'st generation.

1. *Let $\{Y_i\}_{i=1}^{\infty}$ be i.i.d. random variables with values in $\{0\} \cup \mathbb{N}$. Let N be another random variable with values in $\{0\} \cup \mathbb{N}$, and assume that N is independent of $\{Y_i\}_{i=1}^{\infty}$. Let g_Y denote the common generating function of the random variables $\{Y_i\}_{i=1}^{\infty}$. Define*

$$W = \sum_{k=1}^{N} Y_k.$$

   Show that $g_W = g_N \circ g_Y$.

2. *Assume that $X_0 = 1$. Let $p = \mathbb{P}[X_n = 0 \text{ for some } n]$ be the extinction probability. Let g_Z denote the common generating function of the random variables $\{Z_{n,i}\}_{n,i=1}^{\infty}$. Use part 1 of this problem to show that $p = g_Z(p)$.*

   Hint: *Show that when $X_0 = 1$, $g_{X_n}(z) = \underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}}(z)$. Now we have $g_{X_n}(0) = \mathbb{P}[XN = 0]$, and using a little measure theory, one can show* *that*

$$p = \lim_{n\to\infty} \mathbb{P}[X_n = 0].$$

(You may use this fact without proof.)

3. *What is the extinction probability when* $\mathbb{E}[Z_{n,i}] \leq 1$*?*

4. *Is the extinction probability, zero, one, or between zero and one when* $\mathbb{E}[Z_{n,i}] > 1$*?*

   Hint: *Are generating functions always convex?*

The following problems establish some basic properties of the exponential distribution which will be useful in understanding continuous time Markov chains.

Problem 2 (4 points)**.** *Let S be a memoryless random variable with a cumulative distribution function F such that $F(0) = 0$. Assume that F is differentiable. Show that $S \sim$* Exponential(λ) *for some rate $\lambda > 0$. (Recall that a random variable is said to be memoryless if $\mathbb{P}[S > t + s|S > t] = \mathbb{P}[S > s]$ for any $s, t \in [0, \infty)$.)*

Extra Credit: *Prove the same thing without assuming that F is differentiable. Note that you will not need to make any assumptions whatsoever about F, except that $F(0) = 0$.*

Note: *Recall that a random variable G has the geometric distribution with parameter p iff*

$$\mathbb{P}[G = k] = (1 - p)^{(k-1)} p \text{ for all } k \in \mathbb{N}.$$

Observe that the geometric distribution is also memoryless in a certain sense: If G is geometric, then for all $m, n \in \{0\} \cup \mathbb{N}$,

$$\mathbb{P}[G > m + n|G > n] = \mathbb{P}[G > m].$$

However, the geometric distribution is not *memoryless in the sense of this class, because the above relation does not hold for all $m, n \in [0, \infty)$, only integer valued m, n. Think about what would happen for $n = m = 1.1$ to see why this is true.*

Problem 3 (2+2+2 points)**.** *Let $S \sim$* Exponential(α) *and $T \sim$* Exponential(β) *be independent exponential random variables.*

1. *What is the distribution of $S \wedge T$?*

2. *What is the probability that $S \leq T$?*

3. *Show that the two events $\{S < T\}$ and $\{S \wedge T \geq t\}$ are independent.*

Here is a variation of the coupling argument that yields an estimate of the rate of convergence to equilibrium.

Problem 4 (2+3 points)**.** *Let P be an irreducible and aperiodic stochastic matrix with invariant distribution π. Suppose that for some distribution ν and constant $\alpha > 0$,*

$$P_{ij} \geq \alpha \nu_j \text{ for all } i, j.$$

Let λ be a probability distribution.

1. *Let Q be a stochastic matrix. Define a process W_n inductively as follows: Draw W_0 from the distribution λ. Now suppose that W_n has been computed. To compute W_{n+1}, first draw a trial B_{n+1} from the Bernoulli distribution with success probability α. If $B_{n+1} = 1$, then draw W_{n+1} from ν. Otherwise, if $B_{n+1} = 0$, draw W_{n+1} with the distribution $\mathbb{P}[W_{n+1} = j] = Q_{W_n j}$. Assume that all of these various trials and draws are independent. Show that for some stochastic matrix Q, we have $W_n \sim \mathrm{Markov}(\lambda, \mathrm{P})$.*

2. *Let $\{X_n\}_{n=0}^{\infty} \sim \mathrm{Markov}(\lambda, \mathrm{P})$. Show using a version of the coupling argument that*

$$|\mathbb{P}[X_n = i] - \pi_i| \leq (1 - \alpha)^n$$

   for all $i \in \mathbb{E}$.

   Hint: *Define a coupled chain roughly as follows: Let $X_0 \sim \lambda$ and $Y_0 \sim \pi$. With probability α at each time step m, couple the chains by drawing a state i from the distribution ν, setting $X_{m+1} = Y_{m+1} = i$, and demanding $X_n = Y_n$ for all $n \geq m+1$. With the remaining probability $(1-\alpha)$, before coupling has occurred, draw X_{m+1} and Y_{m+1} independently according to the matrix Q from Part 1. Now mimic the coupling argument in the text.*

Finally, here is an exercise that I would solve using the ergodic theorem.

Problem 5 (4 points)**.** *Let $\{X_n\}_{n=0}^{\infty}$ be an irreducible Markov chain with invariant distribution π on the state space E. Let $\{Y_n\}_{n=0}^{\infty}$ be the chain obtained by observing X_n, when it is in $J \subset E$, cf. Homework 2.2, Problem 2. Show that Y_n is positive recurrent, and find a formula expressing its invariant distribution in terms of π.*