Homework 6.1
Stat 605, Fall 2018

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Due Thursday, October 18

Recall the continuous mapping theorem:

Theorem 1. *Let $f : \mathbb{R}^d \to \mathbb{R}^q$ be a continuous function. Let $\{X_n\}_{n=1}^{\infty}$, X be random variables on the same space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in $\mathbb{R}^d$.*

1. *If $X_n \xrightarrow{\text{as}} X$, then $f(X_n) \xrightarrow{\text{as}} f(X)$.*
2. *If $X_n \xrightarrow{\text{P}} X$, then $f(X_n) \xrightarrow{\text{P}} f(X)$.*

The continuous mapping theorem also includes a statement for convergence in distribution:

Problem 1 (2 points)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}^q$ be a continuous function. Let $\{X_n\}_{n=1}^{\infty}$, X be random variables taking values in $\mathbb{R}^d$. (It is not necessary to assume in this case that all random variables are defined on the same probability space.) Show that if $X_n \xrightarrow{\mathscr{D}} X$, then $f(X_n) \xrightarrow{\mathscr{D}} f(X)$.* Hint: *Recall that a composition of continuous functions is continuous.*

Recall that sums and products of measurable functions are measurable. To prove this, we first showed that the Cartesian product (f, g) of a pair of measurable maps from $(\Omega, \mathscr{F})$ to $(\mathbb{R}, \mathscr{B})$ is measurable as a map from $(\Omega, \mathscr{F})$ to $(\mathbb{R}^2, \mathscr{B}^2)$. We then proved that if $H : \mathbb{R}^d \to \mathbb{R}$ is continuous and f is a Borel measurable function with values in $\mathbb{R}^d$, $H \circ f$ is Borel measurable with values in $\mathbb{R}$. It follows that sums and products of measurable functions are measurable, since the sum and product are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Similarly, one might want to show that if $X_n \xrightarrow{\text{P}} X$ and $Y_n \xrightarrow{\text{P}} Y$, then $X_n + Y_n \xrightarrow{\text{P}} X + Y$. This follows from the continuous mapping theorem and the result of the following problem:

Problem 2 (3 points)**.** *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}^d$ valued random variables, and let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}^q$ valued random variables defined on the same probability space as $\{X_n\}_{n=1}^{\infty}$. Prove that if $X_n \xrightarrow{\text{P}} X$ and $Y_n \xrightarrow{\text{P}} Y$, then $(X_n, Y_n) \xrightarrow{\text{P}} (X, Y)$.*

Note: *The X_n's, Y_n's, and (X_n, Y_n)'s are random vectors. By convention, the definition of convergence in probability for random vectors uses the Euclidean* *norm. That is,* $X_n \xrightarrow{\mathrm{P}} X$ *iff for every* $\varepsilon > 0$, $\mathbb{P}[\|X_n - X\| > \varepsilon] \to 0$ *as* $n \to \infty$, *where* $\|\cdot\|$ *is the Euclidean norm. One can show that the choice of norm does not matter as long as the random vectors take values in a finite dimensional space, but please use the Euclidean norm when working out this problem.*

Problem 3 (2+2 points)**.**

1. *Prove or give a counterexample: If* $X_n \xrightarrow{\mathscr{D}} X$ *and* $Y_n \xrightarrow{\mathscr{D}} Y$*, then* $(X_n, Y_n) \xrightarrow{\mathscr{D}} (X, Y)$.
2. *Prove or give a counterexample: If* $X_n \xrightarrow{\mathscr{D}} X$ *and* $Y_n \xrightarrow{\mathscr{D}} Y$*, then* $X_n + Y_n \xrightarrow{\mathscr{D}} X + Y$.

Problem 4 (3 points)**.** *Let* $\{X_i\}_{i=1}^{\infty} \subset L^1(\mathbb{P})$ *be i.i.d. random variables with* $\mathbb{E}[X_i] = \mu$*. Similarly, let* $\{Y_i\}_{i=1}^{\infty} \subset L^1(\mathbb{P})$ *be i.i.d. with* $\mathbb{E}[Y_i] = \nu > 0$*. Prove that*

$$\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} Y_i} \xrightarrow{\text{as}} \frac{\mu}{\nu}.$$

Note: *Please be sure to say something about what happens if the denominator is zero.*

Problem 5 (5 points)**.** *Let* $X_0 = (0, 1) \in \mathbb{R}^2$*, and define a sequence of* $\mathbb{R}^2$ *valued random variables inductively by letting* X_{n+1} *be drawn from the uniform distribution on the disk of radius* $|X_n|$ *centered at the origin. To be more precise, let* $X_{n+1}/|X_n|$ *be uniformly distributed on the disk of radius* 1 *and independent of* $X_1, \dots X_n$*. Prove that*

$$n^{-1} \log|X_n| \xrightarrow{\text{as}} c$$

and compute c.