Homework 5
Stat 697U, Spring 2019

Instructor: Brian Van Koten

Due Wednesday, February 27

The following problem is a version of a famous result called the renewal theorem. It appears as a homework problem in Norris's book.

Problem 1 (3+3 points). *Let $\{Y_n\}_{n=0}^{\infty}$ be independent, identically distributed random variables with values in $\mathbb{N}$. Suppose that the set of integers*

$$\{n : \mathbb{P}[Y_1 = n] > 0\}$$

has greatest common divisor 1. Write $\mu = \mathbb{E}[Y_i]$.

1. *Show that the process*

$$X_n := \inf\{m \geq n : m = Y_1 + \cdots + Y_k \text{ for some } k \geq 0\} - n$$

is a homogeneous Markov chain.

2. *Determine $\lim_{n\to\infty} \mathbb{P}[X_n = 0]$, and show that*

$$\lim_{n\to\infty} \mathbb{P}[n = Y_1 + \cdots + Y_k \text{ for some } k \geq 0] = \frac{1}{\mu}.$$

As Norris says, it is traditional to think of the random variables Y_n as the lifetimes of a lightbulb which is replaced every time it burns out. The conclusion of the renewal theorem is that when n is large, the probability of replacing the lightbulb at time n is about $1/\mu$.

Problem 2 (1 point). *Give a counterexample showing why the hypothesis*

$$gcd\{n : \mathbb{P}[Y_1 = n] > 0\} = 1$$

is needed in the renewal theorem. That is, give an example of a sequence of i.i.d. random variables $\{Y_n\}_{n=0}^{\infty}$ with values in $\mathbb{N}$ so that

$$\lim_{n\to\infty} \mathbb{P}[n = Y_1 + \cdots + Y_k \text{ for some } k \geq 0]$$

either does not exist or does not equal $1/\mu$.

The next problem is a reading assignment. Please feel free to ask questions in class or during office hours if you find the reading confusing.

Problem 3. *Read Section 1.9 on time reversal in Norris's book. A copy is available for free at* `http://www.statslab.cam.ac.uk/~james/Markov/`.

Suppose that you want to compute an expectation with respect to some distribution $\pi \in [0,1]^E$. Of course, if E has only a few elements, this is easy to do: For any $f : E \to \mathbb{R}$, we have

$$\mathbb{E}_\pi[f] = \sum_{i \in E} \pi_i f_i.$$

However, if E is large, it may not be feasible to compute the sum directly. (The situation is usually even more desperate in the case where $E = \mathbb{R}^d$ for d large. We will discuss this case eventually, but for now we must stick to discrete spaces.)

Markov chain Monte Carlo (MCMC) is an alternative method of computing expectations. The basic idea is to simulate a Markov chain with invariant distribution π. Assuming that the ergodic theorem holds, we have

$$\mathbb{E}_\pi[f] = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f(X_t),$$

and therefore to estimate $\mathbb{E}_\pi[f]$ it will suffice to compute the average of $f(X_t)$ over a long trajectory of the chain. For many important problems, this is a more efficient approach than computing the sum directly.

Usually, detailed balance is the key to constructing an appropriate chain with invariant distribution π, although there are exceptions. The following problem introduces the Metropolis–Hastings method, which is one version of MCMC using detailed balance.

Problem 4 (3 points)**.** *Let $\pi \in [0,1]^{\mathbb{Z}}$ be a distribution, and assume that $\pi_i > 0$ for all $i \in \mathbb{Z}$. Typically, in applications of MCMC, one would only know π up to a constant multiple, so suppose that we can only compute entries of $p \in [0,\infty)^{\mathbb{Z}}$ where $p = Z\pi$ for some $Z \geq 0$. Now let $P \in [0,1]^{\mathbb{Z}\times\mathbb{Z}}$ be a stochastic matrix, for example the transition probability of a random walk. Assume that $P_{ij} > 0$ if and only if $P_{ji} > 0$. Define a process X_n as follows:*

1. *Let $X_0 = 0$.*
2. *Given X_n, draw a random* proposal *$\bar{X}_{n+1}$ with the distribution*

$$\mathbb{P}[\bar{X}_{n+1} = j] = P_{X_n, j}.$$

Accept *the proposal with probability*

$$a(X_n, \bar{X}_{n+1}) = 1 \wedge \frac{p_{\bar{X}_{n+1}} P_{\bar{X}_{n+1} X_n}}{p_{X_n} P_{X_n \bar{X}_{n+1}}},$$

and reject *the proposal otherwise. That is, set*

$$X_{n+1} = \begin{cases} \bar{X}_{n+1} & \textit{with probability } a(X_n, \bar{X}_{n+1}), \ \textit{and} \\ X_n & \textit{with probability } 1 - a(X_n, \bar{X}_{n+1}). \end{cases}$$

Show that π *is in detailed balance with the transition matrix of the Markov chain* X_n*.*

Here is another reading assignment. Again, feel free to ask questions.

Problem 5. *Read about the Ehrenfest urn model on page 64 of Brémaud's book. Then read the first part of Section 2.3 up to the discussion of Newton's Law of Cooling. You can read about cooling too, but it uses generating functions, which we haven't yet discussed.*

Problem 6 (3 points)**.** *Find a formula for the invariant distribution of the Ehrenfest urn model. Does it remind you of anything?*