Homework 4.1
Stat 697U, Spring 2019

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Due Wednesday, February 20

Problem 1 (2 points). *Let $(V, \mathcal{E})$ be a finite, undirected, unweighted graph. Here, V denotes the set of vertices, and $\mathcal{E}$ is the set of edges connecting the vertices. To be precise, $\mathcal{E}$ is a set of unordered pairs of elements of V. For each vertex $i \in V$, we define the* valency *v_i of i to be the number of edges having i as an element.*

Corresponding to each graph is a random walk on the vertices V with the transition probability

$$p_{ij} = \begin{cases} \frac{1}{v_i} & \text{if } (i,j) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\pi_i = \frac{v_i}{\sum_{j \in V} v_j}$ is an invariant distribution for the random walk on the graph.

Here is a related problem from Norris's book:

Problem 2 (2+2+2 points). *A particle moves on the eight vertices of a cube. At each step the particle is equally likely to move to any of the three adjacent vertices, independently of its past motions. Let i be the initial vertex occupied by the particle and o the vertex opposite i. (For example, if the cube is $C = \{0,1\}^3$, and $i = (0,0,0)$, then $o = (1,1,1)$.) Calculate each of the following quantities:*

1. *The expected number of steps until the particle returns to i.*
2. *The expected number of visits to o until the first return to i.*
3. *The expected number of steps until the first visit to o.*

Note: *If prefer, you may instead consider a knight starting from one corner of a chessboard instead of a particle on a cube. Assume that at each time step the knight makes one legal move uniformly at random. I don't think it is much harder to answer all the same questions in this case. For example, one may ask how many moves it takes on average for the knight to return to the corner, how many visits to the opposite corner before returning to the initial corner, etc.*

Problem 3 (3 points)**.** *Let P be an irreducible stochastic matrix on a* finite *state space E. Show that the invariant distribution π of P is unique, that $\pi_i > 0$ for all $i \in E$, and that every state $i \in E$ is positive recurrent.* Note: *Don't overthink this problem. I just want you to recall the basic theorems from class, and think about what they mean in the finite case. You may wish to use Theorem 1.7.6 in Norris, which we will discuss in class on Monday.*

Like the last problems on the previous homeworks, the following problems develop properties of Markov chains from the perspective of linear algebra.

Problem 4 (2 points)**.** *Let P be a finite, irreducible, stochastic matrix. Show that all eigenvalues λ of P have $|\lambda| \leq 1$ and that the eigenvalue 1 has multiplicity one.* Hint: *Use the last problem together with the problems on the last homework. Note that the last problem only implies that there is a unique* nonnegative *eigenvector with eigenvalue one.*

Problem 5 (1 point)**.** *Give an example of a finite, irreducible, stochastic matrix P having multiple eigenvalues λ so that $|\lambda| = 1$.*

Here is a method for computing an invariant distribution numerically. It is important to note that although the invariant distribution is a left eigenvector of P, it would be totally inappropriate to use numerical methods for computing eigenvectors, e.g. the QR-algorithm, to compute the invariant distribution. (The method based on QR-factorization below is *not* the QR-algorithm.) The issue is that the eigenvlue corresponding to the invariant distribution is known to be one, and need not be calculated. This simplifies the situation dramatically.

Problem 6 (5 points)**.** *Let $P \in \mathbb{R}^{n \times n}$ be a finite, irreducible stochastic matrix. Any real square matrix M admits a decomposition of the form*

$$M = QR,$$

where Q is orthogonal and R is upper triangular. (This fact is extremely useful in numerical linear algebra, since there are simple, stable algorithms for computing Q and R. Because both orthogonal and triangular matrices are easy to invert, once Q and R are known it is a simple matter to solve equations like $Mx = b$.)

Let I denote the identity matrix, and e the vector of all ones. Define $A = I - P$. Suppose that $A = QR$ for Q orthogonal and R upper triangular. Show that

$$R = \begin{pmatrix} U & -Ue \\ \mathbf{0} & 0 \end{pmatrix}$$

for some nonsingular upper triangular matrix U and that the last column q of Q is an invariant measure for P. Thus, to compute the invariant distribution, it suffices to compute the QR-factorization of A.