Homework 3.2
Stat 605, Fall 2018

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Due Thursday, September 27

Problem 1 (5 points)**.** *Let X be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ and assume that X takes values in the natural numbers. Prove that*

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} \mathbb{P}[X > k].$$

Problem 2. *Let $X : (\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B})$ be a random variable. Show that if F_X is continuous, then $F_X(X)$ has the uniform distribution on $(0,1)$. That is, if $y \in [0,1]$, then $\mathbb{P}(F_X(X) \leq y) = y$.*

Problem 3 (5 points)**.** *Suppose that $X : (\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B})$ is a random variable with continuous density f and that $\mathbb{P}(\alpha \leq X \leq \beta) = 1$. Let $g : (\alpha, \beta) \to \mathbb{R}$ be a strictly increasing differentiable function. Prove that $g(X)$ has density*

$$f_{g(X)}(x) = \begin{cases} \frac{f(g^{-1}(x))}{g'(g^{-1}(x))} & \textit{for } x \in (\alpha, \beta) \\ 0 & \textit{otherwise.} \end{cases}$$

Problem 4 (3 points)**.** *A random variable $X : (\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B})$ is said to be normally distributed if it has density*

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Let X be normally distributed. Compute the density of $\exp(X)$. Note: *This is called the lognormal distribution.*

Problem 5 (3 points)**.** *Let X be normally distributed. Compute the density of X^2.* Note: *This is called the χ^2 distribution.*

Problem 6 (3 points)**.** *A random variable $X : (\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B})$ has the exponential distribution with parameter $\lambda > 0$ if it has density*

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & \textit{if } x \geq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Let δ_0 be the probability measure on $\mathbb{R}$ defined by

$$\delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{otherwise.} \end{cases}$$

Define the probability measure

$$\mathbb{P} = \frac{1}{2}\delta_0 + \frac{1}{2}E_\lambda.$$

where E_λ is the distribution of an exponential random variable with parameter λ. Compute the distribution function of $\mathbb{P}$.