Homework 3.1
Stat 697U, Spring 2019

Instructor: Brian Van Koten

Due Wednesday, February 13

The following problems are taken more or less from Norris's book, Sections 1.2 and 1.5. You will probably want to use two theorems which we did not have a chance to cover this week:

Theorem 1. *Every recurrent class is closed.*

Theorem 2. *Every finite closed class is recurrent.*

We will discuss these theorems next Monday. Feel free to use them without proof on this homework.

Problem 1 (3 points)**.** *Show that any Markov chain on a finite state space has at least one closed communicating class.*

Problem 2 (2 points)**.** *Give an example of a Markov chain with no closed communicating class.*

Problem 3 (1+1+1 points)**.** *Consider the stochastic matrix*

$$P = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

1. *What are the communicating classes of the corresponding Markov chain.*
2. *Which classes are closed?*
3. *Which classes are recurrent and which are transient?*

On the last homework, there was a problem relating eigenvectors of transition matrices and invariant distributions of Markov chains on a finite state space. Recall that a probability vector $v \in [0,1]^E$ is said to be invariant for a transition matrix $P \in [0,1]^E$ if $\{X_n\}_{n=0}^{\infty} \sim \text{Markov}(v, P)$ implies that $X_n \sim v$ for all $n \geq 0$. (Here, $X_n \sim v$ means $v_i = \mathbb{P}[X_m = i]$ for all $i \in E$ and $m \geq 0$.)

By Theorem 1.1.3 in Norris (see your notes from the first lecture), if $\{X_n\}_{n=0}^\infty \sim$ Markov(λ, P), then for any $m \geq 0$,

$$\mathbb{P}[X_m = i] = (\lambda^{\mathrm{t}} P^m)_i \text{ for all } i \in E.$$

Therefore, any invariant distribution of a Markov chain with transition matrix P is a left eigenvector of P with eigenvalue one.

It is easy to see that since the rows of P sum to one, the vector $\mathbf{1}$ whose entries are all ones is a *right* eigenvector of P with eigenvalue one. This implies that there always exists a *left* eigenvector w with eigenvalue one. However, it is not immediately clear that this left eigenvector is an invariant distribution. The issue is that the entries of w may have different signs, in which case there is no scalar α so that αw is a probability vector. (Recall that probability vectors have nonnegative entries that sum to one!)

In fact, with a little bit of extra work, one can show that there exists a left eigenvector with nonnegative entries. This is equivalent with the existence of an invariant distribution.

Problem 4 (2 points)**.** *For any vector $v \in [0,1]^E$, define the ℓ^1-norm of v by*

$$\|v\|_1 = \sum_{i \in E} |v_i|.$$

Let $P \in [0,1]^{E \times E}$ be a stochastic matrix. Show that for any $w \in [0,1]^E$ with $\|w\|_1 < \infty$,

$$\|w^{\mathrm{t}} P\|_1 \leq \|w\|_1.$$

If you like, you may assume that E is finite instead of merely countable, but this is not necessary.

Note: *In general, any vector norm $\|\cdot\|$ induces a matrix norm called the* operator norm *by*

$$\|M\| := \max_{v \neq 0} \frac{\|Mv\|}{\|v\|}.$$

Thus, in this problem, you must show $\|P^{\mathrm{t}}\|_1 \leq 1$.

Problem 5 (2 points)**.** *For $\eta \in \mathbb{R}^E$, define the positive and negative parts $\eta^+, \eta^- \in [0,\infty)^E$ of η by*

$$\eta_i^+ := 0 \vee \eta_i = \max\{0, \eta_i\}$$

and

$$\eta_i^- := -(0 \wedge \eta_i) = -\min\{0, \eta_i\}.$$

Observe that

$$\eta = \eta^+ - \eta^-.$$

Show that $\eta = \eta^+ - \eta^-$ is the minimal decomposition of η as a difference of nonnegative vectors. That is, prove that if $\mu, \nu \in [0,\infty)^E$ are nonnegative vectors *so that* $\eta = \mu - \nu$, *then* $\mu \geq \eta^+$ *and* $\nu \geq \eta^-$. *(Here, for vectors* v *and* w, $v \geq w$ *means* $v_i \geq w_i$ *for all* $i \in E$.*)*

Note: *There is a famous theorem in probability and measure theory called the Hahn–Jordan decomposition. Here, you are proving a special case.*

Problem 6 (3 points)**.** *Let* $P \in [0,1]^{E\times E}$ *be a stochastic matrix, and suppose that* $\eta^{\mathrm{t}} P = \eta^{\mathrm{t}}$ *for some* $\eta \in [0,1]^E$ *with* $\eta \neq 0$. *Show that*

$$(\eta^+)^{\mathrm{t}} P = (\eta^+)^{\mathrm{t}} \text{ and } (\eta^-)^{\mathrm{t}} P = (\eta^-)^{\mathrm{t}}.$$

In other words, if there exists a left eigenvector of P *with eigenvalue one, then there exists a nonnegative left eigenvector of* P *with eigenvalue one.*

Hint: *Use the previous two problems. Also, observe that* P *is a* positive *operator, i.e. if* v *is a nonnegative vector then* $v^{\mathrm{t}} P$ *is also nonnegative.*

You have now proved that invariant distributions exist for Markov chains on a finite state space! If you did all the proofs the way I imagine, then you used finiteness only in showing that a left eigenvector exists and in showing that any nonnegative left eigenvector can be normalized as a probability distribution.