Homework 2.2
Stat 697U, Spring 2019

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Due Wednesday, February 6

Problem 1 (2 points each). *Let X_n be a HMC with state space $E \subset \mathbb{R}$. Which of the following are stopping times for X_n?*

1. $H^A + 7$ *for* $H^A = \inf\{n \geq 0; X_n \in A\}$.
2. *The first time at which X_n attains its maximum for $n = 1, \ldots, 7$:*

$$N_{max} = \inf\{1 \leq n \leq 7; X_n = \max_{k=1,\ldots,7} X_k\}.$$

3. $T \wedge S = \min\{S, T\}$ *for S, T stopping times.*
4. $T \vee S = \max\{S, T\}$ *for S, T stopping times.*
5. *The seventh hitting time of the state $j \in E$:*

$$T_7 = \inf\{n \geq 0; \#\{0 \leq m \leq n; X_m = j\} = 7\}.$$

6. $n \in \mathbb{N}$, *a deterministic time.*

Problem 2 (3+3 points). *Let $X_n \sim$ Markov(λ, P) be a HMC with state space E. Let $A \subset E$.*

1. *For $i, j \in E$, define*

$$h_i^j = \mathbb{P}[X_{T^A} = j | X_0 = i],$$

   Where $T^A := \inf\{n \geq 1 : X_n \in A\}$ is the first passage time to A. Use first-step analysis to find a system of linear equations solved by the probabilities h_j^i.

   Extra Credit: *Show that the h_i^j's are the minimal nonnegative solution of this system of equations.*

2. *Suppose that we can observe the process X_n only when it visits A. Our observations are then a process Y_k, where Y_k is the value of the original process at the time of the k'th visit to A. That is,*

$$Y_k = X_{T_k},$$

where T_k is the k'th passage time to A. (To be precise, we define $T_1 = T^A$ and $T_k = \inf\{n \geq T_{k-1} + 1; X_n = i\}$ for $k > 1$.)

Assume that $\mathbb{P}[T_k < \infty] = 1$ for all $k \in \mathbb{N}$. Show that Y_k is a Markov chain with transition probabilities

$$\mathbb{P}[Y_{k+1} = j | Y_k = i] = h_i^j$$

for all $i, j \in A$.

Problem 3 (3 + 1 points)**.** *Let $X_n \sim$ Markov(λ, P). Assume that the state space E is finite.*

1. *Show that there exists a left eigenvector of P with eigenvalue* 1. Hint: *First, show that there exists a right eigenvector of P with eigenvalue* 1. *To find such an eigenvector, remember that P is a stochastic matrix.*

2. *A probability vector $v \in [0, 1]^E$ is called an* invariant distribution *for P if $X_n \sim$ Markov(v, P) implies that $X_m \sim v$ for all $m \geq 0$, i.e. $\mathbb{P}[X_m = i] = v_i$ for all $i \in E$ and all $m \geq 0$. Does what you have just proved in part 1 have anything to do with the existence of an invariant distribution? Does it imply the existence of an invariant distribution?*