Homework 12.1
Stat 605, Fall 2018

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Due Monday, December 12

The following problem is a famous result called Wald's equation. The point of assinging it is that the technique of proof is different from the proof of the very similar result on the last homework. In particular, as far as I am aware, conditional expectations are not useful here. (I would be happy for one of you to show me that I'm wrong, of course.) On the other hand, one proof does use formulas like $E[N] = \sum_{m=1}^{\infty} \mathbb{P}[N \geq m]$, which you have seen both in the proof of the LLN and also the L^p maximum inequality.

Problem 1 (Optional, highly recommended)**.** *Let X_n be a sequence of i.i.d. random variables, and let N be a stopping time for the filtration $\mathscr{F}_m = \sigma(X_1, \dots, X_m)$. Assume that $\mathbb{E}[|X_i|] < \infty$ and $\mathbb{E}[N] < \infty$. Let $S_m = X_1 + \cdots + X_m$. Show that $\mathbb{E}[S_N] = \mathbb{E}[N]\mathbb{E}[X_1]$.*

Let $\{\xi_i^n\}_{i,n=1}^{\infty}$ be i.i.d. nonnegative integer-valued random variables. Define the *Galton–Watson process* by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0, \\ 0 & \text{if } Z_n = 0. \end{cases}$$

When Galton and Watson proposed this process, they were concerned with the extinction of aristocratic family names in England. In their model, Z_n was the number of males in the n'th generation of some population, and the ξ_i^n were the number of sons of each male.

Let $\mu = \mathbb{E}[\xi_i^n]$, and define $\mathscr{F}_n = \sigma(\{\xi_i^m; i \geq 1, 1 \leq m \leq n\})$. (That is, let $\mathscr{F}_n$ be generated by all the ξ_k^ℓ's with upper index between 1 and n.)

Problem 2 (3 points)**.** *Show that Z_n/μ^n is a martingale with respect to $\mathscr{F}_n$.*

Since Z_n/μ^n is a nonnegative supermartingale, it must converge a.s. It is easy to compute the limit in some cases.

Problem 3 (2 points)**.** *Show that if $\mu < 1$, then with probability one $Z_n = 0$ for sufficiently large n. That is, show that*

$$\mathbb{P}[\{\omega \in \Omega : \exists N(\omega) \text{ so that } Z_n(\omega) = 0 \text{ for all } n \geq N(\omega)\}] = 1.$$

Thus, $Z_n \xrightarrow{\text{as}} 0$. Hint: Z_n only takes integer values.

Problem 4 (2 points)**.** *If* $\mu = 1$ *and* $\mathbb{P}[\xi_i^m = 1] < 1$*, then with probability one* $Z_n = 0$ *for all* n *sufficiently large.* Hint: Z_n only takes integer values.

For $\mu > 1$, it is not so easy to compute the limit.