Homework 11.1
Stat 605, Fall 2018

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Due Thursday, December 6

Problem 1 (3 points)**.** *Let* $X_1, \dots, X_n$ *be independent with* $\mathbb{E}[X_i] = 0$ *and* $\operatorname{var}(X_i) = \sigma_i^2 < \infty$*. Define*

$$S_n = \sum_{k=1}^{n} X_k \text{ and } s_n^2 = \sum_{k=1}^{n} \sigma_k^2.$$

Show that $S_n^2 - s_n^2$ *is a martingale with respect to the filtration* $\mathscr{F}_n = \sigma(X_1, \dots, X_n)$*.*

The original "martingale" was a betting strategy. To keep things concrete, let S_n be the simple random walk defined by $S_0 = 0$ and

$$S_n = S_{n-1} + X_n,$$

where X_i are i.i.d. with $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$. Following the notation in Durrett's book (and from class), the "martingale" corresponds to the bet sequence $H_1 = 1$ and

$$H_n = \begin{cases} 2H_{n-1} & \text{if } X_{n-1} = -1, \text{ and} \\ 1 & \text{if } X_{n-1} = 1 \end{cases}$$

for $n \geq 2$. That is, you double the bet every time you lose. The "martingale" may seem like a great strategy, since if you lose k times and then win, your accumulated winnings since the previous win are

$$-1 - 2 - 2^2 - \cdots - 2^{k-1} + 2^k = 1.$$

Problem 2 (4+2+1 points)**.** *Let* X_n *be a martingale with respect to* $\mathscr{F}_n$*.*

1. *Let* τ *be a stopping time with respect to* $\mathscr{F}_n$*. Assume that* τ *is finite, i.e.* $\mathbb{P}[\tau < \infty] = 1$*. Assume that* X_n *is bounded, i.e. for some* $C > 0$*,* $|X_n(\omega)| \leq C$ *for all* n *and almost all* ω*. Show that* $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$*.*

   Note: *This is an example of an* optional stopping theorem*. There is an entire section devoted to such theorems in Chapter 5 of Durrett's book.*

2. *Give an example of a martingale X_n and a finite stopping time τ with $\mathbb{E}[X_\tau] \neq \mathbb{E}[X_0]$.*

   Hint: *Think about the original "martingale."*

3. *Is the original "martingale" ever a good strategy in practice?*

   Note: *I guess the only reasonable answer is no, but the question is otherwise open ended. I'm not looking for a long explanation. A short paragraph would suffice.*

Problem 3 (2+2+2+2 points)**.** *Consider the random walk S_n and sequence of Bernoulli trials X_n defined above. For $a \in \mathbb{Z}$, let*

$$N_a = \inf\{n \in \mathbb{N} : S_n = a\}$$

be the first hitting time of a. Which of the following are stopping times with respect to the filtration $\mathscr{F}_n = \sigma(X_1, \dots, X_n)$?

1. $N_a - 1$.
2. $N_a + 1$.
3. *The first time at which S_n attains its maximum for $n = 1, \dots, 100$:*

$$N = \inf\{n \in \mathbb{N} : n \leq 100, S_n = \max_{k=1,\dots,100} S_k\}.$$

4. $N_a \wedge c = \min\{N_a, c\}$ *for some* $c \in \mathbb{R}$.

The next problem is called the *switching principle* in Durrett's book:

Problem 4 (5 points)**.** *Let X_n^1 and X_n^2 be supermartingales and N a stopping time with respect to the filtration $\mathscr{F}_n$. Assume that $X_N^1 \geq X_N^2$. Show that*

$$Y_n = X_n^1 \mathbf{1}_{N>n} + X_n^2 \mathbf{1}_{N \leq n}$$

is a supermartingale with respect to $\mathscr{F}_n$.

Finally, here is another problem from Durrett's book. I wonder if it might be a bit difficult. I would be happy to give a hint in class, if needed.

Problem 5 (5 points)**.** *Let X_n and Y_n be positive integrable and adapted with respect to $\mathscr{F}_n$. Suppose that $\sum_{n=1}^\infty Y_n < \infty$ a.s. and that*

$$\mathbb{E}[X_{n+1} | \mathscr{F}_n] \leq (1 + Y_n) X_n.$$

Prove that X_n converges a.s. to a finite limit by finding a closely related supermartingale and applying the martingale convergence theorem.