Homework 10.1
Stat 605, Fall 2018

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Due Thursday, November 29

Problem 1 (3 points)**.** *Let X and Y be random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|XY|] < \infty$. Let $\mathscr{G} \subset \mathscr{F}$ be a σ-algebra, and assume that X is $\mathscr{G}$-measurable. Show that $\mathbb{E}[XY|\mathscr{G}] = X\mathbb{E}[Y|\mathscr{G}]$.* Hint: *Use approximation by simple functions and monotone convergence. I have assigned this fundamental result as an exercise because the method of proof is of such great importance that I want to give you more practice.*

Problem 2 (3 points)**.** *Let X and Y be integrable random variables on $(\Omega, \mathscr{F}, \mathbb{P})$. Show that if $\mathbb{E}[X|Y] = \mathbb{E}[X]$, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*

Problem 3 (3 points)**.** *Prove the conditional version of Chebyshev's inequality: If $a > 0$ and $\mathbb{E}[X^2] < \infty$, then*

$$\mathbb{P}(|X| \geq a|\mathscr{F}) \leq a^{-2}\mathbb{E}(X^2|\mathscr{F}).$$

Problem 4 (3 points)**.** *Let X be a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathbb{E}[X^2] < \infty$, and let $\mathscr{H} \subset \mathscr{G} \subset \mathscr{F}$ be a σ-algebras. Show that*

$$\mathbb{E}[(X - \mathbb{E}[X|\mathscr{G}])^2] + \mathbb{E}[(\mathbb{E}[X|\mathscr{G}] - \mathbb{E}[X|\mathscr{H}])^2] = \mathbb{E}[(X - \mathbb{E}[X|\mathscr{H}])^2].$$

As part of your solution, draw a picture to illustrate why this result is simply a manifestation of the general Pythagorean theorem which holds in any inner product space.

Problem 5 (3 points)**.** *Let $\{X_i\}_{i=1}^{\infty}$ be integrable, i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{var}(X_i) = \sigma^2$. Let N be a natural number valued random variable independent of $\{X_i\}_{i=1}^{\infty}$. Compute $\mathbb{E}[\sum_{i=1}^{N} X_i]$ and $\mathrm{var}(\sum_{i=1}^{N} X_i)$.* Hint: *Condition of the value of N and use the tower law.*

Problem 6 (3 points)**.** *Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. Bernoulli trials with success probability p. Let N_k be the number of trials required to observe k consecutive successes. That is, define*

$$N_k = \min\{n \in \mathbb{N}; X_{n-k+1} = 1, X_{n-k+2} = 1, \ldots, X_n = 1\}.$$

Compute $\mathbb{E}[N_k]$. Hint: *Condition on N_{k-1} to derive a recurrence relation for the N_i's.*

Sometimes, we will say that a sequence of random variables is a martingale without referring to a particular filtration. In that case, the filtration $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$ is implied. The following problem shows that this convention is reasonable:

Problem 7 (3 points)**.** *Let* $\{X_n\}_{n=1}^{\infty}$ *be a martingale with respect to the filtration* $\mathscr{G}_n$*. Let* $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$*. Show that* $\mathscr{F}_n \subset \mathscr{G}_n$ *and* X_n *is a martingale with respect to* $\mathscr{F}_n$*.*