Homework 1
Stat 697U, Spring 2019

Instructor: Brian Van Koten

Due Wednesday, January 30

Recall the Markov property

$$\mathbb{P}[X_{n+1} = i_{n+1} | X_n = i_n, \ldots, X_0 = i_0] = \mathbb{P}[X_{n+1} = j | X_n = i].$$

The Markov property may be extended in many ways—the following problem gives one obvious extension:

Problem 1 (2 points). *Let X_n be a Markov chain. (Do not assume that X_n is homogeneous, only that the Markov property holds.) Prove that for any $n, m \in \mathbb{N}$ and $i_0, \ldots, i_{n+m} \in E$,*

$$\begin{aligned}\mathbb{P}[X_{n+m} &= i_{n+m}, X_{n+m-1} = i_{n+m-1}, \ldots, X_{n+1} = i_{n+1} | X_n = i_n, \ldots, X_0 = i_0] \\ &= \mathbb{P}[X_{n+m} = i_{n+m}, X_{n+m-1} = i_{n+m-1}, \ldots, X_{n+1} = i_{n+1} | X_n = i_n].\end{aligned}$$

The next problem was left as an exercise in class:

Problem 2 (2 points). *Let $\{X_n\}_{n=0}$ be a process taking values in a countable set E, and assume that for some probability vector $\lambda \in [0,1]^E$ and stochastic matrix $P \in [0,1]^{E \times E}$, we have*

$$\mathbb{P}[X_0 = i_0, \ldots, X_N = i_N] = \lambda_{i_0} P_{i_0 i_1} \ldots P_{i_{N-1} i_N}.$$

Prove that $X_n \sim$ Markov(λ, P).

The following problem comes the book *Markov Chains* by J.R. Norris:

Problem 3 (1+1+1+1 points). *Let $\{Z_n\}_{n=0}^{\infty}$ be i.i.d. Bernoulli random variables with $\mathbb{P}[Z_i = 0] = p$ and $\mathbb{P}[Z_i = 1] = 1 - p$. Define $S_n = Z_0 + \cdots + Z_n$. Which of the following processes is a Markov chain?*

1. $A_n = S_n$
2. $B_n = Z_n$
3. $C_n = S_0 + \cdots + S_n$
4. $D_n = (S_n, S_0 + \cdots + S_n)$

For each process that is a Markov chain, find its transition matrix. For each process $X_n \in \{A_n, B_n, C_n, D_n\}$ that is not a Markov chain, find a pair of states i and j so that $\mathbb{P}[X_{n+1} = i | X_n = j, X_{n-1} = k]$ depends on k.