Homework 1.2
Stat 605, Fall 2018

Instructor: Brian Van Koten

Due Thursday, September 13

Problem 1. *Each $x \in [0,1]$ has a binary representation of the form*

$$x = 0.\omega_1\omega_2\omega_3 \cdots = \sum_{i=1}^{\infty} \omega_i 2^{-i},$$

where $\omega = (\omega_1, \omega_2, \dots) \in \{0,1\}^{\mathbb{N}}$. We observe that these binary representations are not exactly unique, since for example

$$\frac{1}{2} = 0.1 = 0.011111 \cdots = \sum_{i=2}^{\infty} 2^{-1}.$$

To remedy this minor difficulty, by convention, one never allows binary representations to terminate with an infinite sequence of consecutive 1's. For convenience, we make one exception, adopting the representation

$$1 = 0.1111111\dots$$

With this convention, binary representations are unique, and we may define a map $f : [0,1] \to \{0,1\}^{\mathbb{N}}$ by

$$f(0.\omega_1\omega_2\omega_3\dots) = (\omega_1, \omega_2, \omega_3, \dots).$$

1. *Prove that f is a measureable map from $([0,1], \mathscr{B})$ to $\big(\{0,1\}^{\mathbb{N}}, \sigma(\mathscr{C})\big)$, where $\mathscr{B}$ denotes the Borel σ-algebra and $\mathscr{C}$ is the collection of cylinder sets defined in the previous homework assignment.* Hint: *Recall our lemma from class: A function f is a measureable map from $(\Omega, \mathscr{F})$ to $(U, \sigma(\mathscr{J}))$, if for every $B \in \mathscr{J}$, $f^{-1}(B) \in \mathscr{F}$.*

2. *Let $\mathbb{P}$ be the uniform distribution on $[0,1]$. Show that the function $Q : \sigma(\mathscr{C}) \to \mathbb{R}$ by $Q(B) = \mathbb{P}(f^{-1}(B))$ is a probability measure and that*

$$Q(\mathscr{C}_{i_1,\dots,i_M}) = 2^{-M}$$

for any cylinder set $\mathscr{C}_{i_1,\dots,i_M}$.

Problem 2. *Let $\mathscr{B}$ denote the Borel σ-algebra on $\mathbb{R}$, and let $\mathbb{Q}$ denote the rational numbers. Define*

$$\mathscr{A} = \{(-\infty, x] : x \in \mathbb{Q}\}.$$

Show that $\mathscr{A}$ generates $\mathscr{B}$; that is, show that

$$\sigma(\mathscr{A}) = \mathscr{B}.$$

Problem 3. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability triple.*

1. *Let "$\backslash$" denote the set difference operation, so*

$$A \backslash B = A \cap B^c.$$

   Prove that

$$\mathbb{P}(A \backslash B) = \mathbb{P}(A) - \mathbb{P}(A \cap B).$$

   Hint: *You don't have any tool at your disposal except countable additivity of measures.*

2. *(Monotonicity) Let $A \subset B$. Show that*

$$\mathbb{P}(A) \leq \mathbb{P}(B).$$

3. *(Finite Subadditivity) Let $A_1, \ldots, A_n \in \mathscr{F}$. Show that*

$$\mathbb{P}(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n \mathbb{P}(A_i).$$

   Hint: *Try for a concise proof, perhaps involving "$\backslash$".*

4. *(Inclusion-Exclusion Principle) Let $A_1, \ldots, A_n \in \mathscr{F}$. Show that*

$$\mathbb{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots A_n),$$

   where for example $\sum_{i<j}$ means to sum over all ordered pairs (i, j) with $i < j$. Hint: *One approach would use induction starting with $n = 2$.*

5. *(Bonferroni Inequality) Let $A_1, \ldots, A_n \in \mathscr{F}$. Show that*

$$\mathbb{P}(\cup_{i=1}^n A_i) \geq \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j).$$