Homework 1.1
Stat 605, Fall 2018

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Due Thursday, September 13

Consider an infinite sequence of coin flips, and recall the notation introduced in class: Let

$$\Omega = \{\mathrm{H}, \mathrm{T}\}^{\mathbb{N}}$$

be the outcome space. For each $i \in \mathbb{N}$ define the random variable

$$X_i : \Omega \to \mathbb{R} \text{ by } X_i(\omega) = \begin{cases} 1 & \text{if } \omega_i = \mathrm{H}, \text{ and} \\ 0 & \text{if } \omega_i = \mathrm{T}, \end{cases}$$

and for each $N \in \mathbb{N}$ define

$$S_N : \Omega \to \mathbb{R} \text{ by } S_N(\omega) = \sum_{i=1}^{N} X_i(\omega).$$

Let $\mathscr{C}$ be the collection of cylinder sets; that is, sets of the form

$$\mathscr{C}_{i_1,\dots,i_M} = \{i_1\} \times \cdots \times \{i_N\} \times \{\mathrm{H}, \mathrm{T}\} \times \{\mathrm{H}, \mathrm{T}\} \times \cdots \subset \Omega$$

for some $M \in \mathbb{N}$ and $i_1, \dots, i_M \in \{\mathrm{H}, \mathrm{T}\}$.

Problem 1. *Fix $k, M \in \mathbb{N}$. Define*

$$D := \left\{ \omega \in \Omega : \text{ for all } N \geq M, \left| S_N(\omega)/N - \frac{1}{2} \right| < \frac{1}{k} \right\}$$

Let $\mathscr{F}$ be a σ-algebra of subsets of Ω, and assume that $\mathscr{C} \subset \mathscr{F}$. Prove that $D \in \mathscr{F}$. Hint: *Show that D is a countable intersection of sets in $\mathscr{C}$.*

Problem 2. *Let*

$$E := \left\{ \omega \in \Omega : \lim_{N \to \infty} S_N(\omega)/N = \frac{1}{2} \right\}.$$

Let $\mathscr{F}$ be a σ-algebra of subsets of Ω, and assume that $\mathscr{C} \subset \mathscr{F}$. Prove that $F \in \mathscr{F}$. Hint: *Write E in terms of events like D using only countable unions and intersections.*