Homework 7

Math 651
Fall 2019

Due Wednesday, November 13, 2019

Problem 1 (1+5 points)**.**

1. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be continuously differentiable. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^m$, a norm on $\mathbb{R}^n$, and also the induced operator norm on $\mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$. Assume that for some $L > 0$, we have*

$$\|f'(x)\| \leq L \text{ for all } x \in \mathbb{R}^m.$$

Show that f is Lipschitz with constant L.

2. *Show that*

$$\lim_{N\to\infty} \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{N \text{ square roots}} = 2.$$

Hint: *Use the contraction mapping theorem.*

Definition 1. Let x_n be a sequence in a normed vector space V, and assume that $\lim_{n\to\infty} x_n = z$. The sequence x_n is said to be *convergent of order p* if for some $p \geq 1$ and $C > 0$, $\|x_{n+1} - z\| \leq C\|x_n - z\|^p$.

As usual, all other things being equal, a higher order of convergence is better than a lower order of convergence. Recall that for the stationary Newton method, with β, γ, and r defined as in class,

$$\|x_{n+1} - z\| \leq \beta\gamma r\|x_n - z\|,$$

so stationary Newton is convergent of order $p = 1$. In the following problem, you will show that the Newton method is of order $p = 2$.

Problem 2 (6 points)**.** *Let $U \subset \mathbb{R}^n$ be open and convex, and let $f : U \to \mathbb{R}^n$ be continuously differentiable. Assume the following:*

(a) There exists $\gamma > 0$ so that $\|f'(x) - f'(y)\| \leq \gamma\|x - y\|$ for all $x, y \in U$.

(b) $f'(z)$ is invertible for all $z \in U$, and there exists $\beta > 0$ so that $\|f'(z)^{-1}\| \leq \beta$ for all $z \in U$.

(c) *We have* $\|f'(x_0)^{-1}(c - f(x_0))\| < \frac{1}{2\beta\gamma}$.

(d) *For* $r := 2\|f'(x_0)^{-1}(c - f(x_0))\|$, *we have* $B_r(x_0) \subset U$.

We have seen in class that these conditions guarantee the existence of $z \in B_r(x_0)$ *solving* $f(z) = c$. *Now let*

$$\Phi_c(x) := x + f'(x)^{-1}(c - f(x))$$

be the iteration map corresponding to Newton's method. Show that

$$\|\Phi_c(x) - z\| \leq \frac{\beta\gamma}{2}\|x - z\|^2,$$

i.e. prove that Newton's method converges quadratically.

Finding roots of polynomials is (usually) an ill-conditioned problem. The following exercise gives examples.

Problem 3 (2+3+2+1 points)**.**

1. *Let* $p(x) = (x-1)^2$. *Suppose that we wish to solve the equation* $p(z) = 0$. *For each* $c \geq 0$, *let* z_c *be a solution of* $p(z_c) = c$. *(Of course, there are two solutions for* $c > 0$. *It doesn't matter which one you choose to be* z_c.*) Show that*
$$\hat{\kappa} := \lim_{\delta \to 0} \sup_{c \in [0,\delta]} \frac{|z_c - z_0|}{|c - 0|} = \infty.$$

2. *Let*
$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$
be a polynomial of degree n. *Assume that* z *is a root of* p *with* $p'(z) \neq 0$. *Let* $i \in \{1, \ldots, n\}$. *What are the absolute and relative condition numbers of the root* z *with respect to perturbations of the coefficient* a_i*?*

   In case the question is not clear, here is a more verbose restatement: Define the perturbed polynomial
$$p_\delta(x) = p(x) + x^i \delta.$$
   One can show that for some small ε, *there exists a differentiable function* $\eta : [-\varepsilon, \varepsilon] \to \mathbb{R}$ *with* $\eta(0) = z$ *so that* $p_\delta(\eta(\delta)) = 0$. *(This is a consequence of the inverse function theorem, which is itself more-or-less a consequence of our analysis of the stationary Newton method.) What is the condition number of the solution* $\eta(\delta)$*?*

3. *Consider the famous Wilkinson polynomial*
$$q(x) = \prod_{i=1}^{20}(x - i).$$

Plot the first 200 *iterates generated by Newton's method (not the stationary Newton method) starting from* $x_0 = 100$*. Does it look like Newton's method is converging? One can show that in exact arithmetic, Newton's method started from* $x_0 = 100$ *converges monotonically to* 20*. But your Newton method may not converge so nicely.*

4. *Do the observations above tell you anything about how you should compute the eigenvalues of a matrix?*

Problem 4 (up to 6 points extra credit)**.** *One can show that the root finding problem for the Wilkinson polynomial is ill-conditioned. See* `https://www.maa.org/sites/default/files/pdf/upload_library/22/Chauvenet/Wilkinson.pdf`*. However, when I ran the example above with my own implementation of the Wilkinson polynomial and its derivatives, it worked perfectly! Read Wilkinson's article linked above, and find me an example that actually fails to converge. You may have to increase the number of factors, substituting* 20 *for* 100 *in the definition of* q*, for instance.*