Homework 6

Math 651
Fall 2019

Due Friday, November 1, 2019

In this homework, you will complete the proof of backwards stability for the solution of $Ax = b$ via QR-factorization. Let $\tilde{x}$ be the approximate solution computed using Householder QR-factorization and back substitution. Recall that for some $\Delta Q, \Delta R \in \mathbb{R}^{n \times n}$ with

$$\frac{\|\Delta Q\|_2}{\|\tilde{Q}(A)\|_2} = O(\varepsilon_m) \text{ and } \frac{\|\Delta R\|_2}{\|\tilde{R}(A)\|_2} = O(\varepsilon_m),$$

we have

$$(\tilde{R}(A) + \Delta R)\tilde{x} = (\tilde{Q}(A)^{\mathrm{t}} + \Delta Q^{\mathrm{t}})b. \tag{1}$$

Problem 1 (2+2+2+2 points).

1. *Let $\tilde{Q}(A), \tilde{R}(A)$ be the Householder QR-factorization of A computed in floating point arithmetic. Show that*

   $$\frac{\|\tilde{R}(A)\|_2}{\|A\|_2} \leq 1 + O(\varepsilon_m),$$

   where the function $O(\varepsilon_m)$ above does not depend on A.

   Hint*: If Q is orthogonal, what is $\|Q\|_2$? Also, Householder QR-factorization is backwards stable.*

2. *Let ΔQ be as in (1). Show that*

   $$\frac{\|E_2\|_2}{\|A\|_2} := \frac{\|\Delta Q \tilde{R}(A)\|_2}{\|A\|_2} \leq O(\varepsilon_m),$$

   where the function $O(\varepsilon_m)$ above does not depend on A. (Recall that $E_2 = \Delta Q \tilde{R}(A)$ is one of the error terms defined in class.)

3. *Let ΔR be as in (1). Show that*

   $$\frac{\|E_3\|_2}{\|A\|_2} := \frac{\|\tilde{Q}(A)\Delta R\|_2}{\|A\|_2} \leq O(\varepsilon_m),$$

   where the function $O(\varepsilon_m)$ above does not depend on A.

4. Show that

$$\frac{\|E_4\|_2}{\|A\|_2} := \frac{\|\Delta Q \Delta R\|_2}{\|A\|_2} \le O(\varepsilon_m^2),$$

where the function $O(\varepsilon_m^2)$ above does not depend on A.

As a consequence of the first problem and the argument started in class, we have the error estimate

$$\frac{\|\tilde{x}(\tilde{R}(A), \tilde{Q}(A)^{\mathrm{t}} \circledast b) - x\|_2}{\|x\|_2} \le (\kappa_2(A) + o(1))O(\varepsilon_m),$$

where again the function $O(\varepsilon_m)$ does not depend on A.

Problem 2 (up to 8 points, extra credit)**.** *Suppose that we want estimates in a norm other than $\|\cdot\|_2$. Answer some or all of the following questions: Is it still possible to get error estimates where the $O(\varepsilon_m)$ term does not depend on A? What do these estimates look like? Does the $O(\varepsilon_m)$ term depend on the dimension of A?*