Homework 4

Math 651
Fall 2019

Due October 11, 2019

The following fact will be extremely useful when we start discussing partial differential equations.

Problem 1 (2+2 points). *Let $\|\cdot\|$ denote both a norm on $\mathbb{R}^n$ and also the induced operator norm on $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$. Let $M \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$. Suppose that for some $\theta > 0$, we have*

$$\|Mx\| \geq \theta \|x\|$$

for all $x \in \mathbb{R}^n$.

1. *Show that M is invertible.* Hint: *What is the kernel of M?*
2. *Show that $\|M^{-1}\| \leq 1/\theta$.*

The next problem is to prove the relationship between the condition number of a matrix and the distance from that matrix to the set of singular matrices.

Problem 2 (1+1+2+2+2 points).

1. *Let $v \in \mathbb{R}^n$. Prove that*

$$\|v\|_2 = \max_{\substack{w \in \mathbb{R}^n \\ \|w\|_2 = 1}} \langle v, w \rangle.$$

   Hint: *Use the Cauchy inequality.*

2. *Let $v, z \in \mathbb{R}^{n \times 1}$. (That is, let v and z be column vectors.) Show that*

$$\|vz^{\mathrm{t}}\|_2 = \|v\|_2 \|z\|_2.$$

   Hint: *Note that vz^{t} is a square matrix. Use the first part of this problem to find its operator norm.*

3. *Let $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ be invertible. Show that*

$$\min\{\|\Delta A\|_2 : A + \Delta A \text{ singular}\} \geq \frac{1}{\|A^{-1}\|_2}.$$

   Hint*: Think about the theorem that we proved using the matrix geometric series.*

4. *Let $v \in \mathbb{R}^{n\times 1}$ be such that (a) $\|v\|_2 = 1$, and (b) $\|A^{-1}v\|_2 = \|A^{-1}\|_2$. That is, let v be a unit vector for which the operator norm of A^{-1} is attained. Show that*

$$A - \frac{v(A^{-1}v)^{\mathrm{t}}}{\|A^{-1}\|_2^2}$$

is singular.

5. *Combine the results above to show that*

$$\frac{1}{\|A\|_2\|A^{-1}\|_2} = \frac{\min\{\|\Delta A\|_2 : A + \Delta A \text{ singular}\}}{\|A\|_2}.$$

The problem below shows that subtraction is sometimes an ill-conditioned problem.

Problem 3 (3 points)**.** *Consider the problem $s : \mathbb{R}^2 \to \mathbb{R}$ by $s(x_1, x_2) = x_1 - x_2$. Equip both the data space, $\mathbb{R}^2$, and the solution space, $\mathbb{R}$, with the $\|\cdot\|_\infty$ norm. Show that*

$$\kappa(s, x) = \frac{2\max\{x_1, x_2\}}{|x_1 - x_2|}.$$

Hint: *It is easy to explicitly compute the Jacobian matrix for this simple problem.*

Problem 4 (2 points)**.** *Let $f : \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ by*

$$f(M) = M^3.$$

Show using the product rule that f is differentiable and

$$f'(M)(H) = HMM + MHM + MMH.$$

Note: *This is kind of like $\frac{d}{dx}x^3 = 3x^2$.*

Problem 5 (1+3+3 points, extra credit)**.** *Let $J : \{M \in \mathbb{R}^{n\times n}, M \text{ invertible}\} \to \mathbb{R}^{n\times n}$ by $J(M) = M^{-1}$. In this problem, you may assume without proof that whenever M is invertible, $J(M)$ is differentiable at M.*

1. *Let $G : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be differentiable at M. Define $K : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by*

$$K(M) = MG(M).$$

   Use the product rule to show that

$$K'(M)(H) = HG(M) + M[G'(M)(H)].$$

2. *Show that*

$$J'(M)(H) = -M^{-1}HM^{-1}.$$

   Hint: *Use implicit differentiation. We know that $MJ(M) = I$ for all M. Now differentiate both sides, observe that the derivative of the right hand side is zero, and solve for $J'(M)(H)$.*

3. *Let* $\|\cdot\|$ *be the operator norm on* $\mathbb{R}^{n\times n}$ *induced by some norm on* $\mathbb{R}^n$*. Use the above formula for the derivative of* J *to show that*

$$\kappa(J, M) \leq \|M\|\|M^{-1}\|.$$

(If you want even more extra credit, show that in fact $\kappa(J, M) = \|M\|\|M^{-1}\|$*.)*