Homework 3

Math 651
Fall 2019

Due October 4, 2019

The next problem concerns what would happen if p were less than one in the definition of the p-norm.

Problem 1 (2+1+2 points)**.** For any $p \in (0, \infty)$, $x \in \mathbb{R}^n$, define the p-distance

$$(x)_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}.$$

(Note that $(\cdot)_p = \|\cdot\|_p$ for $p \in [1, \infty)$, but here we allow $p < 1$.)

1. Draw graphs of the unit balls

   $$B_p := \left\{ x \in \mathbb{R}^2 : (x)_p \leq 1 \right\} \subset \mathbb{R}^2$$

   in the p-distance for $p \in \{\frac{1}{2}, 1, 2, \infty\}$.

2. A subset U of a vector space is said to be *convex* if and only if $x, y \in U$ implies $\alpha x + (1 - \alpha) y \in U$ for any $\alpha \in [0, 1]$. Prove that the unit ball in any normed vector space is convex.

3. Show that $(\cdot)_p$ is not a norm for any $p \in (0, 1)$.

You already have a formula expressing the ℓ^2 operator norm in terms of the singular values of a matrix. This is not so nice, because to calculate the norm you have to calculate the singular values, and this is not so easy. For many purposes, it is better to consider ℓ^1 or ℓ^∞ operator norms, since these can be evaluated explicitly, as you will now demonstrate.

Problem 2 (1+2+2 points)**.** Let $M \in \mathbb{R}^{n \times n}$ be a matrix. Let $M_{i:}$ denote the i'th row of M, and let $M_{:j}$ denote the j'th column.

1. Let $v \in \mathbb{R}^n$. Show that

   $$\|v\|_1 = \sup\{\langle w, v \rangle : w \in \mathbb{R}^n, \|w\|_\infty = 1\}.$$

   Also, show that

   $$\|v\|_\infty = \max\{\langle w, v \rangle : w \in \mathbb{R}^n, \|w\|_1 = 1\}.$$

2. Prove that
$$\|M\|_\infty = \max_{i=1,\dots,n} \|M_{i:}\|_1.$$
3. Prove a similar formula for $\|M\|_1$.

Definition 1. Two norms $\|\cdot\|$ and $|\cdot|$ on a vector space V are said to be *topologically equivalent* if and only if there exist constants $c, C > 0$ so that

$$c|u| \leq \|u\| \leq C|u|$$

for all $u \in V$.

As the language suggests, if two norms are topologically equivalent, they share all topological properties. In particular, if a sequence converges with respect to one of the norms, then it converges with respect to the other. It is a famous theorem that all norms on a *finite*-dimensional vector space are equivalent. (This is certainly not the case for *infinite*-dimensional vector spaces.) I was going to have you prove that this is true, but then I decided it was more important to have you work out the following problem, which is closely related.

Problem 3 (2+2+2 points)**.** Let $n \in \mathbb{N}$.

1. Find the greatest constant $c_n > 0$ and smallest constant $C_n > 0$ so that
$$c_n\|v\|_1 \leq \|v\|_\infty \leq C_n\|v\|_1$$
for all $v \in \mathbb{R}^n$.
2. Find the greatest constant $k_n > 0$ and smallest constant $K_n > 0$ so that
$$k_n\|v\|_1 \leq \|v\|_2 \leq K_n\|v\|_1.$$
3. Define the L^1 norm on $C([0,1])$ by
$$\|f\|_1 = \int_0^1 |f|\, dx.$$
Give an example of a sequence of functions f_n in $C([0,1])$ so that $\|f_n\|_1 = 1$ for all n but $\lim_{n\to\infty}\|f_n\|_\infty = \infty$. Observe that the existence of such a sequence implies that the L^1 and L^∞ norms on $C([0,1])$ are *not* equivalent.

You should have noticed that some of the equivalence constants above depend on n. In fact, some of the constants tend to zero or infinity as n increases. In numerical analysis we are almost always considering the limit of large n, and so the theorem on equivalence of norms on a finite-dimensional space is not often useful.

The following problem concerns the loose ends in our proof of the theorem on condition numbers.

Problem 4 (2+1 points)**.** Let $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ be an invertible linear operator. Let $\|\cdot\|$ denote both a norm on $\mathbb{R}^n$ and its induced operator norm on $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$. Let $\tilde{A} = A + \delta A$ be a perturbation of A.

1. Show that if $\|A^{-1}\|\|\delta A\| < 1$, then $\tilde{A}$ is invertible and

$$\|\tilde{A}^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|\delta A\|}.$$

2. Suppose that $\tilde{A}$ is invertible. Let $\tilde{b}, u \in \mathbb{R}^n$, let y solve $\tilde{A}y = \tilde{b} + u$, and let $\tilde{x}$ solve $\tilde{A}\tilde{x} = \tilde{b}$. Show that

$$\|y - \tilde{x}\| \leq \|\tilde{A}^{-1}\|\|u\|.$$