Homework 2

Math 651
Fall 2019

Due September 27, 2019

Recall that a spline of order m with knots $\{x_0, \dots, x_n\}$ is a function $s : \mathbb{R} \to \mathbb{R}$ such that

1. $s \in C^{m-1}$, and
2. $s|_{[x_i, x_{i-1}]} \in \mathscr{P}^m$ for all $i = 1, \dots, n$.

Let $\mathscr{S}^m$ denote the set of all order m splines with a given set of knots. In class, I presented a heuristic argument based on counting degrees of freedom to suggest that the dimension of $\mathscr{S}^m$ is $n + m$. Now I would like you to give a rigorous proof of this fact by constructing an explicit basis for the space of splines.

Problem 1 (5 points)**.** Define

$$x_+^m := \begin{cases} x^m, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

for $m \in \mathbb{N}$. Show that the $m + n$ functions

$$\begin{aligned} u_k(x) &:= (x - x_0)^k, && \text{for } k = 0, \dots, m, \\ v_k(x) &:= (x - x_k)_+^m, && \text{for } k = 1, \dots, n-1, \end{aligned}$$

are a basis for $\mathscr{S}^m$. That is, show that these functions are linearly independent and that every spline in $\mathscr{S}^m$ can be expressed as a linear combination of them.

Recall that the cubic spline interpolating polynomial has an optimality property. For $u, v \in C^2([a, b])$, define the H^2-seminorm and inner product by

$$\begin{aligned} (u, v)_{H^2} &:= \int_a^b u'' v'' \, dx \\ |u|_{H^2} &:= (u, u)_{H^2}^{\frac{1}{2}} = \left(\int_a^b (u'')^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Now fix $f \in C^2$. It turns out that the cubic spline $S_3 f$ interpolating f is exactly the minimizer of $|\cdot|_{H^2}$ over the set of all functions $g \in C^2$ sharing the values of f at the knots. In class, I outlined a proof of optimality based on the Pythagoras theorem. I would now like for you to fill in the details.

Problem 2 (4+1 points)**.** Let s be the cubic spline interpolating polynomial of the function $f \in C^2$ on the knots $\{x_0, \dots, x_n\}$ with the natural boundary condition.

1. Show that $(f - s, s)_{H^2} = 0$, i.e. the error $f - s$ is orthogonal to s in H^2. *Hint:* Write the inner product as a sum of integrals over subintervals of the form $[x_{-1}, x_i]$. Use integration by parts twice on each of the integrals in the sum. Try to show that the various boundary terms cancel.

2. Use part 1 of this problem to prove

$$|s|_{H^2}^2 + |f - s|_{H^2}^2 = |f|_{H^2}^2,$$

which is a version of the Pythagoras theorem. Since $|f - s|_{H^2}^2 \geq 0$, this proves optimality.

In the usual (Lagrange) interpolation problem, the values of a certain function are specified on a set of nodes. In Hermite interpolation, we assume in addition that the values of the derivative of the function are known at the nodes. The following problems cover the basic facts on Hermite interpolation.

Let $x_0, \dots, x_n$ be a set of $n + 1$ distinct nodes, and let $\{\ell_k : k = 0, \dots, n\}$ be the Lagrange characteristic polynomials associated with these nodes. Define the *Hermite factors*

$$H_k^0(x) := [1 - 2\ell_k'(x_k)(x - x_k)]\ell_k(x)^2 \text{ and } H_k^1(x) := (x - x_k)\ell_k(x)^2.$$

Problem 3 (3 points)**.** Suppose that

$$f(x_k) = y_k \text{ and } f'(x_k) = y_k' \text{ for } k = 0, \dots, n$$

Prove that the polynomial

$$H_n f = \sum_{k=0}^{n} y_k H_k^0 + y_k' H_k^1 \in \mathscr{P}^{2n+1}$$

has $H_n f(x_k) = f(x_k)$ and $(H_n f)'(x_k) = f'(x_k)$ for all $k = 0, \dots, n$.

The following problem will test that you understand the derivation of the error formula for Lagrange polynomial interpolation. I have decided to make this problem extra credit.

Problem 4 (8 points, extra credit)**.** Let $f \in C^{2n+2}$. Let $H_n f$ denote the Hermite interpolating polynomial of f on the nodes $x_0 < \cdots < x_n$. Show that for any $x \in [x_0, x_n]$, there exists $\xi_x \in [x_0, x_n]$ so that

$$f(x) - H_n f(x) = \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} \prod_{j=0}^{n} (x - x_j)^2.$$

Hint: Mimic the proof of the error formula for Lagrange polynomial interpolation.