Each of these two problems is worth 10 points. Do all four problems.

1. Given a smooth function \( z = f(x, y) \) and a point \((x^0, y^0)\) at which \( \nabla f(x^0, y^0) \neq (0, 0) \), consider the level curve that passes through that point, and let \( u \) be a unit vector that makes an angle \( \alpha \) with the tangent line to the level curve. In terms of the angle \( \alpha \) and partial derivatives of \( f \), determine the value of the directional derivative at \((x^0, y^0)\); that is,

\[
\frac{\partial f}{\partial u}(x^0, y^0) = D_u f(x^0, y^0)
\]

**Solution:** If \( u \) makes an angle \( \alpha \) to the tangent line, then it makes an angle \( \pi/2 - \alpha \) to the gradient, which is perpendicular to the tangent line to the level set. Therefore, the directional derivative equals \( \|\nabla f(x^0, y^0)\| \sin \alpha \).

Actually, there is an ambiguity in the problem, in that the sense of the angle \( \alpha \) is not specified. If the angle \( -\alpha \) is used, then the directional derivative gets a minus sign too.

2. Consider the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) defined by the component functions

\[
\begin{align*}
f_1 &= x^2 - 2y \\
f_2 &= 2x^2 - xy \\
f_3 &= 3x^2 y - 2x
\end{align*}
\]

Calculate the derivative (Jacobian) matrix \( Df(x, y) \) at the point \((x, y) = (3, -1)\).

**Solution:** Calculate all the various partial derivatives and assemble them into the \( 3 \times 2 \) the matrix:

\[
Df(x, y) = \begin{bmatrix} 2x & -2 \\ 4x - y & -x \\ 6xy - 2 & 3x^2 \end{bmatrix}
\]

Now evaluate all the matrix entries at the given point.

3. Consider the function \( f(x, y, z) = \sqrt{x^6 + y^3z^3} \).

(a) Show that \( f \) is a “homogeneous function” in the sense that

\[
f(tx, ty, tz) = t^p f(x, y, z) \quad \text{for all} \quad t, \quad \text{and all} \quad x, y, z.
\]
What is the “degree” \( p \) of this homogeneous function \( f \)?

(b) Show that any homogeneous function \( f \) satisfies the identity

\[
x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = pf.
\]

**Solution:** In part (a), just calculate the value of this function at the scaled point \( t(x, y, z) \), and notice that a factor of \( t^3 \) factors out; actually, this is true only for positive \( t \), otherwise there is a minus sign. For part (b), differentiate the equation defining homogeneity with respect to \( t \), and use the chain rule. You get

\[
\frac{d}{dt} f(tx, ty, tz) = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z
\]

and

\[
\frac{d}{dt} \{t^p f(x, y, z)\} = pt^{p-1} f(x, y, z)
\]

Now set \( t = 1 \) (this is the trick), and you obtain the desired identity.

4. (a) Calculate the Taylor expansion of \( f(x) = e^{\sin x} \) about the point \( x = 0 \), up to second order.

(b) Approximate \( e^{\sin 1} \) to an accuracy of 3 decimal places. Can you justify that your error is smaller than \( 10^{-3} \)?

**Solution:** For part (a), you need to calculate the first and second derivatives of \( f \), which are

\[
f'(x) = e^{\sin x} \cos x, \quad f''(x) = e^{\sin x} [\cos^2 x - \sin x]
\]

Evaluating these at \( x = 0 \), produces the Taylor expansion

\[
f(x) = 1 + x + x^2/2 + R_2
\]

It is amusing to note that up to second order, this expansion is the same as for the function \( e^x \) itself; basically, this is a consequence of the fact that \( \sin x = x + O(|x|^3) \). For part (b), you need to estimate the remainder \( R_2(x) \) at the point \( x = 0.1 \). To do this, you need to calculate the third derivative (which is rather long):

\[
f'''(x) = e^{\sin x} \left\{ \cos x [\cos^2 x - \sin x] - 2 \cos x \sin x - \cos x \right\}
\]

\[
= -e^{\sin x} \cos x \sin x [3 + \sin x]
\]

where the second expression results from some simplification. The remainder estimate requires finding an upper bound on \( \max |f'''(x)| \) over \( 0 \leq x \leq 0.1 \). From the above expression, a crude (but quite decent) estimate for this is \( e^{0.1} \times \sin 0.1 \times 4 \), which in turn is less than 0.5. You can get this estimate without a calculator, recalling that \( \sin x \leq x \). Consequently, the desired estimate for the remainder is

\[
|R_2(0.1)| \leq \frac{1}{6} \times 0.5 \times (0.1)^3 \leq 10^{-4}
\]

This means that the second order Taylor expansion gives the value of of \( f(0.1) \) to better than three decimal places. Even cruder estimates are sufficient for the required result.