1. Consider a roll of paper, like a toilet roll. Its cross section is very nearly an annulus of inner radius $R_1$ and outer radius $R_2$. The thickness of the paper is $\tau$. Devise a simple mathematical model that allows you to estimate the total length of the paper on the roll. State and explain any simplifications, assumptions, or approximations you make.

SOLUTION: There is more than one way to approach this basic problem. Here are two natural approaches.

Method 1. View the roll as a series of concentric rings, like those of a tree. The number of “rings” is then $N = (R_2 - R_1)/\tau$. The total length is then

$$L = \sum_{n=1}^{N} 2\pi(R_1 + n\tau) = 2\pi NR_1 + 2\pi\tau\frac{N(N + 1)}{2} \approx 2\pi \frac{(R_2 - R_1)R_1}{\tau} + 2\pi \frac{(R_2 - R_1)^2}{2\tau} = \frac{\pi(R_2^2 - R_1^2)}{\tau}$$

The approximation $N \approx N + 1$ used to get this neat expression is based on $N \gg 1$.

Method 2. View the roll as a continuous spiral, given in polar coordinates by

$$r = f(\theta) = R_1 + \frac{\tau\theta}{2\pi} \quad \text{where} \ \theta \ \text{runs from} \ 0 \ \text{to} \ 2\pi N.$$ 

The total length is then given by the formula for arclength of a polar curve:

$$L = \int_{0}^{2\pi N} \sqrt{f'(\theta)^2 + f(\theta)^2} \, d\theta \approx \int_{0}^{2\pi N} f(\theta) \, d\theta \quad \text{since} \ |f'| \ll f$$

$$= 2\pi NR_1 + \pi\tau N^2 = \frac{\pi(R_2^2 - R_1^2)}{\tau}$$

Notice that this derivation gives exactly the same result.

In fact, the end result is intuitive: One could have guessed it by noticing that the total cross-sectional area of the roll is $A = \pi(R_2^2 - R_1^2)$. So the total length $L$ of the roll must be such that $L \times \tau = A$. In this way it is possible to see the answer immediately.

As far as approximations are concerned, the key is that $\tau$ is small compared to $R_2 - R_1$, and hence that $N$ is large.

2. Calculate the exact period of the nonlinear pendulum. That is, consider equation
governing the angular displacement $\theta = \theta(t)$:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin \theta = 0.$$ 

Nondimensionalize time $t$ so that this equation simplifies to $\frac{d^2\theta}{dt^2} + \sin \theta = 0$. Assume that the pendulum starts in a state of rest with an angular displacement $\alpha$, for any $0 < \alpha < \pi$. Use the following steps to calculate the oscillation period $T = T(\alpha)$.

**SOLUTION:** I went through these steps in detail in class, and so I will only give the results briefly here.

(a). Derive conservation of energy. To do this, multiply the second-order differential equation by $\frac{d\theta}{dt}$ and manipulate to get an equation that relates $\frac{d\theta}{dt}$ to $\theta$, but does not involve $\frac{d^2\theta}{dt^2}$. Can you relate the terms in this equation to kinetic and potential energy?

The crucial identity is:

$$0 = \frac{d\theta}{dt} \cdot \left\{ \frac{d^2\theta}{dt^2} + \sin \theta \right\} = \frac{d}{dt} \left\{ \frac{1}{2} \left( \frac{d\theta}{dt} \right)^2 + (1 - \cos \theta) \right\} = \frac{dE}{dt}$$

in which the curly bracket defines the total energy $E$. Since $E$ is necessarily constant in time, this step reducing the given second-order differential equation to a first-order differential equation for $\theta$ — that is, we have taken the first integral. The constant $E$ is determined by the initial condition that the pendulum is at rest at time $t = 0$ with an angular displacement $\theta(0) = \alpha$. Thus, $E = 1 - \cos \alpha$.

(b). Now solve for $\frac{d\theta}{dt}$ in terms of $\theta$ itself. By using the method for separable first-order DEs, express the half period $T/2$ that the pendulum takes to go from $\theta = -\alpha$ to $\theta = +\alpha$ as an integral with respect to $\theta$. Specifically, get the formula:

$$\frac{T}{2} = \int_{0}^{\alpha} \frac{d\theta}{\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}.$$ 

The separable DE that results is

$$\frac{d\theta}{\sqrt{2(\cos \theta - \cos \alpha)}} = dt$$

The half-angle trig identity expresses these cosines in terms of sine squared. The integral is taken over a quarter period, during which the pendulum swings from 0 to $\alpha$. Be careful to keep track of all the various factors of 2.

(c). But $\alpha$ appears in two places in this integral formula. So it is desirable to recast it in another form by the following tricky substitution: let $\phi$ by a new integration variable, defined by $\sin(\alpha/2) \cdot \sin \phi = \sin(\theta/2)$. Now get the formula

$$T = 4 \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2(\alpha/2) \sin^2 \phi}}.$$ 

This tricky substitution tricky would be very hard to find by yourself. But there is nothing to it beside following the rules for changing variable in an integral. Be sure to express $d\theta$ in terms of $d\phi$, and change the limits of integral appropriately.
(d). Determine the Taylor expansion of $T(\alpha)$ up to order $\alpha^2$ around zero, and interpret the correction term $T(\alpha) - 2\pi$ for finite-amplitude swings of the pendulum.

While this formula is nice, it is not elementary. In fact, it is a so-called complete elliptic integral, which is not reducible any further. But one can proceed with approximations to get a simpler result that is valid if $\alpha$ is not too large. Use the Taylor expansion

$$(1 - x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + O(x^4)$$

for $x$ near 0. Calculate the approximated integrand, and then use the expansion $\sin x = x + O(x^3)$, to simplify the result further. The final result is

$$T(\alpha) = 2\pi \left[ 1 + \frac{\alpha^2}{16} + O(\alpha^4) \right]$$

This formula shows how the period of the pendulum increases as its maximum angular displacement grows. Of course, as $\alpha$ approaches the extreme value $\pi$ the period goes to infinity, and our approximation is no longer valid.

3(a). A sound wave in the air consists of small oscillations in pressure and density (because it consists of rapid expansions and contractions of parcels of air). So it is reasonable to expect that the speed, $c$, of sound in air depends upon the equilibrium air pressure $p$ and density $\rho$ around which the pressure and density fields oscillate. Determine the physical units of $p$ and $\rho$. Use dimensional analysis to establish the following formula:

$$c^2 = \frac{\gamma p}{\rho}, \quad \text{in which } \gamma \text{ is a dimensionless constant.}$$

The value of $\gamma$ cannot be found by this argument; for air $\gamma \approx 1.4$.

(b). Suppose that a spherical “bullet” is shot into the air at high speed (which may be supersonic). The bullet has radius $R$ and travels with velocity $V$. Determine how the drag force, $F_D$, on the bullet depends on the parameters of the problem, assuming that the relevant physical quantities are $R, V, \rho, c$. (Notice that here we let $c$ take the place of $p$ by virtue of part a.) There is one dimensionless function in this formula, which cannot be determined by dimensional arguments; explain what this (arbitrary) function means and how experiments might be used to evaluate it.

**SOLUTION:** This and the next problem use dimensional analysis to get significant results without actually formulating and solving difficult equations of motion. We use the following symbols for the independent physical dimensions: $M=\text{mass}, \ L=\text{length}, \ T=\text{time}$. Pressure is force per unit area, so its units are $[p] = ML^{-1}T^{-2}$; density is mass per unit volume, so its units are $[\rho] = ML^{-3}$. Thus, if the sound speed $c$, whose units are $[c] = LT^{-1}$, is expressible as a monomial in $p$ and $\rho$ — that is, if there exists a physically meaningful formula $c = C\rho^a p^b$ for some dimensionless constant $C$ and some powers $a$ and $b$ — then it is easy to see that $a = 1/2$ and $b = -1/2$. No other powers give a result that is consistent with any system of physical units.

Now consider the high speed sphere traveling through the air. The units involved in this problem are: $[F_D] = MLT^{-2}$, $[V], [c] = LT^{-1}$, $[R] = L$, and $[\rho] = ML^{-3}$. In this problem, there are more determining quantities (4) than units (3). The idea is then to find a formula that nondimensionalizes the drag force $F_D$ in terms of some of these units, but also involves an arbitrary function of some other dimensionless group. The obvious choice is the ratio $\mu = V/c$, which is called the Mach number. This eliminates $c$ in terms of $V$. (Note that we also replaced $p$ by $c$ at the outset.) We seek a formula

$$F_D = \rho^8 V^\gamma R^\delta f(\mu) \quad \text{where } f \text{ is an dimensionless function.}$$
Using dimensional analysis, the only choice of powers is \( q = 1, r = 2, s = 2 \). And thus we obtain the desired result.

In the scientific literature this would be written in the form

\[
\frac{F_D}{\rho V^2 R^2} = f\left(\frac{V}{c}\right),
\]

which records the dependence of nondimensionalized drag force on Mach number. Experimental data would be reported in this form, since the dependence of the force on \( \rho \) and \( R \) is no surprise.

A final comment: One could choose to nondimensionalize \( F_D \) by the product \( Vc \) rather than \( V^2 \), or even by \( c^2 \). But this just changes the function \( f \). The one that is given above is the standard choice.

4. Simple model of the expansion of a supernova. When a star explodes in a supernova, there is an initial stage in which it freely expands, but then it enters a stage in which a shock wave forms in the interstellar gas around the star as the blast pushes against the interstellar medium. We wish to model this second stage crudely. Assume that the expanding shock wave is a sphere of radius \( R = R(t) \) that depends on the time \( t \) since the explosion. There are only two other relevant physical quantities in this problem: the total energy \( E \) of the supernova that occurs at \( t = 0, R = 0 \), and the (uniform) density \( \rho \) of the interstellar gas. Using only dimensional analysis, determine how \( R \) depends on \( t, E, \rho \). Also determine how the velocity, \( V = dR/dt \), evolves in time. Do these results seem reasonable?

**SOLUTION:** This problem is very similar to the preceding one. The key assumption is that the radius of the expanding shock wave is determined by \( E, \rho \) and \( t \) entirely. This is a valid assumption for a certain phase of the expansion (from about 1000 to 10,000 years after the supernova explosion). Given this assumption, the problem is a simple one in dimensional analysis because the number of determining quantities equals the number of independent units.

The result is

\[
R = C E^{1/5} \rho^{1/5} t^{2/5}
\]

where \( C \) is a dimensionless constant.

It turns out that \( C \) is close to 1; but dimensional analysis alone cannot determine it.

The expansion therefore is more rapid at small time \( t \) than at later times. Indeed, the corresponding speed of the shock wave is \( V = dR/dt \), and so it varies like \( t^{-3/5} \). This behavior does seem physically reasonable. What is interesting is that dimensional analysis is able to provide the precise exponent \( 2/5 \), while intuition only suggests some power less than one.

5. Old-fashioned GPS. A ship at sea uses foghorns to determine its location, especially its distance from land (=rocks and shoals). Suppose that two emitters of sound (=foghorns) are located on shore at positions \((x_1, y_1)\) and \((x_2, y_2)\). They emit synchronized signals that travel at the speed of sound, \( c \), to the ship, which is at a location \((x, y)\), to be determined. The ship records the arrivals of the signals and deduces the elapsed times, \( t_1 \) and \( t_2 \), from each lighthouse.

**a.** From \( t_1, t_2, c \), determine the location \((x, y)\). Is the solution unique?

**b.** Carry out a sensitivity analysis, which considers errors in the three measurements, say \( \Delta t_1, \Delta t_2, \Delta c \). Determine how errors in the estimated location of the ship, say, \( \Delta x, \Delta y \), depend on the errors of the input measurements. In what geometrical configuration are the location errors the worst?
**SOLUTION:** (a) Given $t_1, t_2, c$, the position $(x, y)$ to be determined satisfies the nonlinear system of 2 equations

\[
(x - x_1)^2 + (y - y_1)^2 = c^2 t_1^2, \quad (x - x_2)^2 + (y - y_2)^2 = c^2 t_2^2.
\]

The easiest way to solve this system for $x$ and $y$ is to subtract the second equation from the first, thereby obtaining a linear equation relating $x$ to $y$; namely,

\[
(x_2 - x_1)(2x) + (y_2 - y_1)(2y) = c^2(t_1^2 - t_2^2).
\]

Using this linear equation, solve for $y = y(x)$ in terms of $x$. Now substitute the result into one or other of the original equations, obtaining a single quartic equation in $x$ alone. Algebraically the result is quite messy; but nonetheless, conceptually, the procedure is straightforward. Notice that there are going to be two solutions, corresponding to a symmetry with respect to the emitters. In practice, you would know which “side” of the emitters you are located, and so you could eliminate the spurious solution.

(b) It is obviously unpleasant to do the sensitivity analysis directly from the messy solution formula obtained in (a). Instead, proceed with this analysis using implicit differentiation. That is, expand the solution corresponding to the perturbed data, $t_1, t_2, c$, to first order in $\Delta x$ and $\Delta y$. Up to first order accuracy, the perturbation of the solution satisfies the linear system of equations

\[
2(x - x_1)\Delta x + 2(y - y_1)\Delta y = 2\epsilon_1, \quad 2(x - x_2)\Delta x + 2(y - y_2)\Delta y = 2\epsilon_2,
\]

in which we introduce the small quantities, $\epsilon_i = c^2 c_1^2 + c^2 t_i\Delta t_i$, for each $i = 1, 2$.

The linear system for $(\Delta x, \Delta y)$ therefore determines the sensitivity properties. The only situation in which small inputs $(\epsilon_1, \epsilon_2)$ will result in large outputs $(\Delta x, \Delta y)$ is when the coefficient matrix for this linear system is nearly singular. That is, when the determinant

\[
\det \begin{pmatrix}
 x - x_1 & y - y_1 \\
 x - x_2 & y - y_2
 \end{pmatrix}
\]

is very small.

Of course, this condition has the interpretation that the two displacement vectors from the two emitters to the receiver are nearly collinear. A further analysis (skipped here) would show that the resulting error is in the angular determination of the receiver position, not its radial distance from the emitters. In a good GPS system the emitters are put far enough apart so that this loss of precision does not occur; in this business it is call the “Geometric Dilution or Precision.”