Visualizing Roots of Cubics (Cardano’s Formula) and other Polynomial Equations.
Solving the Polynomial Equations!

- **Linear Equation:**
  \[ ax + b = 0 \iff x = -\frac{b}{a} \]

- **Quadratic Equation:**
  \[ ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

What if \( \deg f(x) \geq 3 \) ?

- Factor into multiple polynomials.
- \( z^n = 1 \iff z = e^{\frac{i\pi}{n}k} \)
- Etc. etc…

But, can we have a general formula for \( \deg f(x) \geq 3 \)?
\[
\text{Solve}[ax^3 + bx^2 + cx + d = 0, x]
\]

\[
\{x \rightarrow \frac{1}{3a} \sqrt[3]{-\frac{27a^2d + 9abc - 2b^3}{2} + 4 \left(\frac{3ac - b^2}{3\sqrt[3]{2}}\right)^3 - \frac{27a^2d + 9abc - 2b^3}{3}}\}, \{x \rightarrow \frac{1}{3a} \sqrt[3]{-\frac{27a^2d + 9abc - 2b^3}{2} + 4 \left(\frac{3ac - b^2}{3\sqrt[3]{2}}\right)^3 - \frac{27a^2d + 9abc - 2b^3}{3}} + \frac{\sqrt{3}}{2} \left(\frac{3ac - b^2}{3\sqrt[3]{2}}\right)\}, \{x \rightarrow \frac{1}{3a} \sqrt[3]{-\frac{27a^2d + 9abc - 2b^3}{2} + 4 \left(\frac{3ac - b^2}{3\sqrt[3]{2}}\right)^3 - \frac{27a^2d + 9abc - 2b^3}{3}} - \frac{b}{3a}\}, \{x \rightarrow \frac{1}{3a} \sqrt[3]{-\frac{27a^2d + 9abc - 2b^3}{2} + 4 \left(\frac{3ac - b^2}{3\sqrt[3]{2}}\right)^3 - \frac{27a^2d + 9abc - 2b^3}{3}} - \frac{b}{3a}\}, \{x \rightarrow \frac{1}{3a} \sqrt[3]{-\frac{27a^2d + 9abc - 2b^3}{2} + 4 \left(\frac{3ac - b^2}{3\sqrt[3]{2}}\right)^3 - \frac{27a^2d + 9abc - 2b^3}{3}} - \frac{b}{3a}\}\]
Generalization of Cubic Roots
–Cardano’s method

- Method that makes the General solution for Cubic Equation in the shape of:
  \[ x^3 + px + q = 0 \]

- For the general cubic equation with \( x^2 \) term – Tschirnhaus transformation

- It’s actually the Tartaglia’s Theorem!
Tschirnhaus transformation

Cubic Equation \( ax^3 + bx^2 + cx + d = 0 \) \((a \neq 0)\)

Divided both side by \(a\) and substitution

\[
\left(y - \frac{b}{3a}\right)^3 + \frac{b}{a} \left(y - \frac{b}{3a}\right)^2 + \frac{c}{a} \left(y - \frac{b}{3a}\right) + \frac{d}{a} = 0 \quad \left(x = y - \frac{b}{3a}\right)
\]

Expand, and simplify:

\[
y^3 + \frac{3ac - b^2}{3a^2} y + \frac{2b^3 - 9abc + 27a^2d}{27a^3} = 0
\]

\[\Rightarrow y^3 + py + q = 0 \]

Point of this transformation: deleting the \(x^2\) term.
Let \( u + v = y \)

\[
(y^3 + py + q = 0)
\]

\[
(u + v)^3 + p(u + v) + q = 0
\]

\[
\Rightarrow (u^3 + v^3 + q) + (3uv + p)(u + v) = 0
\]

Since this equation must be valid for any variable \( u \) and \( v \) (where \( u + v = y \)),

\[
\begin{cases}
  u^3 + v^3 + q = 0 \\
  3uv + p = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases}
  q = -u^3 - v^3 \\
  -\frac{p}{3} = 3uv
\end{cases}
\]

(Note: Constant \( p \) and \( q \) are the function of variable \( u \) and \( v \) as above – value of variables depends on the constants – this equation is very likely to have a general solution)
Since $u^3 + v^3 = -q$ and $u^3v^3 = -\frac{p^3}{27}$, $u^3$ and $v^3$ are two roots of the quadratic equation

$$x^2 + qx - \frac{p^3}{27} = 0 \implies \begin{cases} x_1 = u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \\ x_2 = v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \end{cases}$$

$$x_1 = (\sqrt[3]{x_1})^3 = u^3 \implies \left(\frac{u}{\sqrt[3]{x_1}}\right)^3 = 1$$

$$\implies \frac{u}{\sqrt[3]{x_1}} = 1, \frac{-1 + \sqrt{3}i}{2} (= \omega), \frac{-1 - \sqrt{3}i}{2} (= \omega^2)$$

$$\implies u = \sqrt[3]{x_1}, \sqrt[3]{x_1} \omega, \sqrt[3]{x_1} \omega^2$$
By the same logic, \( v = \sqrt[3]{x_2}, \sqrt[3]{x_2}\omega, \sqrt[3]{x_2}\omega^2 \)

Remember that \( u^3 + v^3 = -q \) and \( u^3v^3 = -\frac{p^3}{27} \):

\[ \therefore (u, v) = (\sqrt[3]{x_1}, \sqrt[3]{x_2}), (\sqrt[3]{x_1}\omega, \sqrt[3]{x_2}\omega^2), (\sqrt[3]{x_1}\omega^2, \sqrt[3]{x_2}\omega) \]

\[ \therefore x = y - \frac{b}{3a} = u + v - \frac{b}{3a} = \begin{cases} 
\sqrt[3]{x_1} + \sqrt[3]{x_2} - \frac{b}{3a} \\
\sqrt[3]{x_1}\omega + \sqrt[3]{x_2}\omega^2 - \frac{b}{3a} \\
\sqrt[3]{x_1}\omega^2 + \sqrt[3]{x_2}\omega - \frac{b}{3a}
\end{cases} \]
\textbf{Solve} \{ ax^3 + bx^2 + cx + d = 0, x \} \\
\{ \{ x \to \sqrt[3]{\frac{-27a^2d + 9abc - 2b^3}{3^{3/2}a}} + 4 \left( \frac{3ac - b^2}{3a} \right) - \frac{b}{3a} \}, \{ x \to \frac{3^{3/2} (3ac - b^2)}{3a} \} \} \\
\{ \{ x \to \frac{3^2 2^{1/3} a \sqrt{-27a^2d + 9abc - 2b^3} + 4 \left( \frac{3ac - b^2}{3a} \right) - 27a^2d + 9abc - 2b^3}{6^{3/2}a} \}, \{ x \to \frac{(1 + i \sqrt{3}) (3ac - b^2)}{3a} \} \} \\
\{ \{ x \to \frac{3^2 2^{1/3} a \sqrt{-27a^2d + 9abc - 2b^3} + 4 \left( \frac{3ac - b^2}{3a} \right) - 27a^2d + 9abc - 2b^3}{6^{3/2}a} \}, \{ x \to \frac{(1 - i \sqrt{3}) (3ac - b^2)}{3a} \} \} \\
\{ \{ x \to \frac{3^2 2^{1/3} a \sqrt{-27a^2d + 9abc - 2b^3} + 4 \left( \frac{3ac - b^2}{3a} \right) - 27a^2d + 9abc - 2b^3}{6^{3/2}a} \}, \{ x \to \frac{-b}{3a} \} \} \}
Other than Cardano’s Method?

- Vieta’s Method
  - Same approach as Cardano’s method basically, but slightly different in substitution.

- Lill’s Method
  - Using geometry to find the roots of polynomials with any degrees or any coefficients.

Cardano’s method is simplest and most widely used solution until today.
**QUESTION:**
Can we find the discriminant of cubic equation as we did for quadratic equation?

\[ ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ \Rightarrow D = b^2 - 4ac \]

**Answer:** Yes we can! In general, we define the discriminant as follows:

\[ D = \left( \prod_{i<j} (\alpha_i - \alpha_j) \right)^2 \]

Where \( \alpha_i \)'s are the roots of the equation.
Let \( f(x) = ax^3 + bx^2 + cx + d \). Then
\[
\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty \quad \text{when} \quad a > 0
\]
\[
\lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty \quad \text{when} \quad a < 0
\]
f(x) must pass the x-axis at least once (Intermediate Value Theorem); in other words, \( f(x) \) must hold at least one real root.

If \( z = a + bi \) is a solution of \( f(x) \), \( \bar{z} = a - bi \) is the solution as well.

pf) \( f(z) = f(\bar{z}) \)

Three cases:
1) Three different real solutions: \( D > 0 \)
2) A multiple real root (or two real roots): \( D = 0 \)
3) One real and two complex roots: \( D < 0 \)
Use the same transformation!

\[ ax^3 + bx^2 + cx + d = 0 \Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0 \Rightarrow y^3 + py + q = 0 \]

Let \( \alpha, \beta, \) and \( \gamma \) be three solutions of \( y^3 + py + q = 0 \)

\[
\alpha + \beta + \gamma = 0, \quad \alpha\beta + \beta\gamma + \gamma\alpha = p, \quad \alpha\beta\gamma = -n
\]

\[
\Rightarrow (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \gamma^2 - 4\alpha\beta
\]

\[
\therefore D = (\gamma^2 - 4\alpha\beta)(\beta^2 - 4\gamma\alpha)(\alpha^2 - 4\beta\gamma)
\]
\[
= -63\alpha^2\beta^2\gamma^2 - 4(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3) + 16\alpha\beta\gamma(\alpha^3 + \beta^3 + \gamma^3)
\]
\[
= -63p^3 - 4(p^3 + 3q^2) + 48q^2 = -4p^3 - 27q^2
\]
\[
= -27a^2d^2 - 4ac^3 + 18abcd - 4db^3 + b^2c^2
\]

\[
\therefore D = -27a^2d^2 - 4ac^3 + 18abcd - 4db^3 + b^2c^2 \quad (\because a^4 > 0)
\]

Better way: Draw a graph..
How about Quartic Equation?
The solution of Quartic Equations can be generalized as Linear, Quadratic, and cubic.

Ferrari’s solution: Solving $y^4 + ay^2 + by + c = 0$

For general solution of general quartic equation:

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Divide both sides with $a$:

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$$

Substitution: $x = y - \frac{b}{4a}$

$$y^4 + py^2 + qy + r = 0$$

Use Ferrari’s method next.
According to Abel–Ruffini theorem discovered by Niels Abel and Evariste Galois, there is no general algebraic solution to polynomial equations of degree fifth or higher with any coefficients. We must find the solutions to the equations by using factorization or graph. Proof of Abel–Ruffini theorem requires Lie Algebra and Abstract Algebra.
None of the calculator gives the perfect general solution; they either do not give any result or assumes $a$ to be zero.
Thanks!