COMPACTIFICATIONS OF SUBVARIETIES OF TORI

By Jenia Tevelev

Abstract. We study compactifications of subvarieties of algebraic tori defined by imposing a sufficiently fine polyhedral structure on their non-archimedean amoebas. These compactifications have many nice properties, for example any k boundary divisors intersect in codimension k. We consider some examples including $M_{0,n} \subset \overline{M}_{0,n}$ (and more generally log canonical models of complements of hyperplane arrangements) and compact quotients of Grassmannians by a maximal torus.

1. Introduction and statement of results. Let X be a connected closed subvariety of an algebraic torus T over an algebraically closed field k. It is natural to consider compactifications $\overline{X}$ defined as closures of X in various toric varieties $\mathbb{P}$ of T. In this paper we address the following question: how nice could $\overline{X}$ possibly be? Consider the multiplication map

$$\Psi\colon\ T \times \overline{X} \to \mathbb{P}, \quad (t,x) \mapsto tx.$$

Definition 1.1. We call $\overline{X}$ a *tropical compactification* if Ψ is faithfully flat and $\overline{X}$ is proper. A term "tropical" is explained in Section 2.

Theorem 1.2. *Any subvariety X of a torus has a tropical compactification $\overline{X}$ such that the corresponding toric variety $\mathbb{P}$ is smooth. In this case the boundary $\overline{X} \setminus X$ is divisorial and has "combinatorial normal crossings": for any collection $B_1, \ldots, B_r \in \overline{X} \setminus X$ of irreducible divisors, $\cap B_i$ has codimension r. Moreover, if $r = \dim X$ and $p \in B_1 \cap \cdots \cap B_r$ then $\overline{X}$ is Cohen-Macaulay at p.*

We also show that if $\overline{X} \subset \mathbb{P}$ is tropical then $\overline{X}' \subset \mathbb{P}'$ is also tropical for any proper toric morphism $\mathbb{P}' \to \mathbb{P}$. However, it is not known (at least to the author) if there exists a minimal tropical compactification.

The boundary $\overline{X} \setminus X$ does not necessarily have genuine normal crossings.

Definition 1.3. A subvariety X of a torus T is called *schön* if it has a tropical compactification with a smooth multiplication map.

Theorem 1.4. *If X is schön then any of its tropical compactifications $\overline{X} \subset \mathbb{P}$ has a smooth multiplication map, is regularly embedded, normal, has toroidal*

Manuscript received August 22, 2005; revised February 23, 2006.
American Journal of Mathematics 129 (2007), 1087–1104.

singularities, and its log canonical line bundle is globally generated and is equal to the determinant of the normal bundle. Moreover, if $\mathbb{P}' \to \mathbb{P}$ is a proper toric morphism then the corresponding map $\overline{X}' \to \overline{X}$ is log crepant.

Schön hypersurfaces in tori are known as "hypersurfaces nondegenerate with respect to their Newton polytope", introduced and studied by Varchenko [V2], [V1].

Next we discuss one natural class of schön subvarieties of tori of high codimension. Let X be a complement of a hyperplane arrangement,

$$X = \mathbb{P}^r \setminus \{H_1, \ldots, H_n\}. \tag{$\circ$}$$

Assume that the arrangement is essential (not all hyperplanes pass through a point) and connected (does not have any projective automorphisms). $\mathbb{P}^r$ admits an embedding in $\mathbb{P}^{n-1}$ such that the hyperplanes H_i are intersections with coordinate hyperplanes of $\mathbb{P}^{n-1}$. It follows that X is a closed subvariety of a torus $(k^*)^{n-1}$.

THEOREM 1.5. *X is schön and Kapranov's visible contour $\overline{X}_{vc}$ of X [CQ1] is one of its tropical compactifications. The log canonical line bundle of $\overline{X}_{vc}$ is very ample and $\overline{X}_{vc}$ is the log canonical model of X.*

This theorem was essentially proved in [CQ2, Theorem 2.19] but we give another argument for the reader's convenience. We discuss this result and compare the tropical compactification with other known compactifications of complements of hyperplane arrangements in Section 4.

In fact, our proof of existence of tropical compactifications is constructive and blends the following ingredients:

- Generalized Kapranov's visible contours.
- Lafforgue's transversality argument of [Laf, Thm. 4.5].
- Nonarchimedean amoebas.

We will explain generalized visible contours below, everything else is explained in Section 2. Let T_X be the connected component of the normalizer of X in T. The action of T_X on T (and hence on X) is free and the quotient X/T_X is a closed subvariety of a torus T/T_X. We will construct a tropical compactification of X/T_X. Since $X \simeq (X/T_X) \times T_X$, it easily follows that X also has a tropical compactification.

Definition 1.6. This construction depends on a choice of an auxiliary proper toric variety $T \subset \mathbb{P}$. Let $\overline{X}$ be the closure of X in $\mathbb{P}$. For any closed subscheme Y of $\mathbb{P}$, we denote by $[Y] \in \mathrm{Hilb}(\mathbb{P})$ the corresponding point of the Hilbert scheme. Consider the twisted action of T on $\mathrm{Hilb}(\mathbb{P})$:

$$T \times \mathrm{Hilb}(\mathbb{P}) \to \mathrm{Hilb}(\mathbb{P}), \quad t \cdot [Y] = t^{-1}[Y].$$

The normalization of the closure of the T-orbit of $[\overline{X}]$ in $\mathrm{Hilb}(\mathbb{P})$ is called the *Gröbner toric variety* $\mathbb{P}_{Gr}$ of $(X, \mathbb{P})$. It compactifies the torus T/T_X. Let $\overline{X}_{vc}$

be the closure of X/T_X in $\mathbb{P}_{Gr}$. Let $\mathbb{P}_{vc} \subset \mathbb{P}_{Gr}$ be the toric open subset consisting of all T-orbits that intersect $\overline{X}_{vc}$. We consider $\overline{X}_{vc}$ as a closure of X/T_X in $\mathbb{P}_{vc}$ (and not $\mathbb{P}_{Gr}$) to make the multiplication map surjective.

THEOREM 1.7. *$\overline{X}_{vc}$ is a tropical compactification of X/T_X. Moreover,*

$$(\star) \qquad \overline{X}_{vc} = \{[Y] \in \mathbb{P}_{Gr} \mid e \in Y\},$$

where $e \in T$ is the identity element.

The formula $(\star)$ is a direct generalization of Kapranov's visible contour [CQ1], which is a special case when X is a complement of a hyperplane arrangement as in 1.5, and Hilb$(\mathbb{P})$ is just the Grassmannian.

Remark 1.8. Though $\overline{X}_{trop}$ is defined using Hilbert schemes, in fact we use only the closure of the T-orbit of $[\overline{X}]$. This makes $\overline{X}_{trop}$ computable, see Section 2.10. A different method of constructing tropical compactifications is discussed in Remark 3.3.

Any tropical compactification of X/T_X can be viewed as a compact quotient space of X. It turns out that it is related to (and refines) other quotient constructions such as GIT quotients or more precisely their inverse limits, Chow/Hilbert quotients. This relationship is explained in Section 5. We concentrate in particular on tropical quotients of Grassmannians by the maximal torus.

In what follows we assume that the reader is familiar with the basics of the logarithmic Mori theory [KMM].

Acknowledgments. I am grateful to Sean Keel for a lot of help, in particular he suggested Theorem 1.4 to me. This paper is a part of the project [CQ2], [HKT], [KT] on Chow quotients of Grassmannians. I am grateful to Paul Hacking, Misha Kapranov, Laurent Lafforgue, Bernd Sturmfels, David Speyer, Martin Olsson, and David Saltman for several illuminating discussions.

2. Tropical compactifications. Let X be a connected closed subvariety of an algebraic torus T over an algebraically closed field k. A transfinite Puiseux series over k is a formal sum $f(z) = \sum_{q\in\mathbb{Q}} f_q z^q$, where $f_q \in k$ are such that Supp$(f) = \{q\colon f_q \neq 0\}$ is well-ordered (i.e. every subset of it has a minimal element). Such series form a field $\overline{K}$ that is algebraically closed in any characteristic [R]. Let deg: $\overline{K} \to \mathbb{Q}$ be the non-archimedean valuation (the minimal exponent of the series). Let $X(\overline{K}) \subset T(\overline{K})$ be $\overline{K}$-points. The valuation extends to the surjective map

$$\deg\colon\ T(\overline{K}) \to \Lambda^\vee_{\mathbb{Q}},$$

where $\Lambda^\vee$ is the $\mathbb{Z}$-lattice of 1-PS of T and $\Lambda^\vee_{\mathbb{Q}} = \Lambda^\vee \otimes \mathbb{Q}$.

Definition 2.1. [EKL] The *nonarchimedean amoeba* of a pair (X, T) is

$$\mathcal{A} = \deg X(\overline{K}) \subset \Lambda^\vee_{\mathbb{Q}}.$$

By [EKL, Theorem 2.2.5], $\mathcal{A}$ is the underlying set of a fan (maybe not uniquely determined), and more generally, if X is defined over K then $\mathcal{A}$ is the underlying set of a polyhedral complex (maybe not uniquely determined) and is equal to the Bieri–Groves set [BG] of X.

LEMMA 2.2. *Let $\mathcal{F} \subset \Lambda^\vee_{\mathbb{Q}}$ be a rational polyhedral cone such that $\mathcal{F} \cap (-\mathcal{F}) = \{0\}$. Let $\mathbb{P}$ be the corresponding affine toric variety. Then $\mathcal{A}$ intersects the (relative) interior of $\mathcal{F}$ if and only if the closure $\overline{X}$ of X intersects the closed orbit of $\mathbb{P}$.*

Proof. Suppose that $\mathbb{P}$ is smooth, i.e. the rays of $\mathcal{F}$ are spanned by vectors from a basis of $\Lambda^\vee$. We can assume that $\Lambda^\vee$ has a basis $x_1, \ldots, x_n$ and that $\mathcal{F}$ is spanned by $x_1, \ldots, x_d$. We identify $T \simeq (k^*)^n$ and $\mathbb{P} \simeq \mathbb{A}^d \times (k^*)^{n-d}$ using this basis. If $\mathcal{F}$ contains a point of $\mathcal{A}$ in its interior then $X(\overline{K})$ contains a point

$$p(z) = (p_1(z), \ldots, p_n(z)), \tag{1}$$

where z is a uniformizer in K, $\deg p_i(z) > 0$ for $i \leq d$, and $\deg p_i(z) = 0$ for $i > d$. It follows that $\overline{X}(k)$ contains a point $p(0) = (0, \ldots, 0, p_{d+1}(0), \ldots, p_n(0))$ that belongs to the closed orbit. And vice versa, if $\overline{X}$ intersects the closed orbit of $\mathbb{P}$ then one can find a germ of a curve (1) in X such that $\deg p_i(z) \geq 0$ for any i and $(p_1(0), \ldots, p_n(0))$ belongs to the closed orbit. In particular, $\deg p(z)$ belongs to the interior of $\mathcal{F}$.

Now consider the general case. Let $Z \subset \mathbb{P}$ be the closed orbit. Let $\pi\colon \mathbb{P}' \to \mathbb{P}$ be a toric resolution of singularities and let $\mathcal{F} = \bigcup_i \mathcal{F}_i$ be the corresponding subdivision, here $\dim \mathcal{F}_i = \dim \mathcal{F}$ for any i. Then $\mathcal{A}$ intersects the (relative) interior of $\mathcal{F}$ if and only if there exists i such that $\mathcal{A}$ intersects the (relative) interior of a face $\mathcal{G} \subset \mathcal{F}_i$ and $\mathcal{G}$ is not on the boundary of $\mathcal{F}$. By the previous special case, $\mathcal{A}$ intersects the (relative) interior of a face $\mathcal{G} \subset \mathcal{F}_i$ if and only if the closure of X in $\mathbb{P}'$ intersects the closed orbit of the affine chart corresponding to $\mathcal{G}$. Therefore, $\mathcal{A}$ intersects the (relative) interior of $\mathcal{F}$ if and only if the closure of X in $\mathbb{P}'$ intersects $\pi^{-1}(Z)$. The latter happens if and only if the closure of X in $\mathbb{P}$ intersects Z since π is proper. □

PROPOSITION 2.3. *Let $\mathbb{P}$ be a toric variety of T (not necessarily proper) with the fan $\mathcal{F} \subset \Lambda^\vee \otimes \mathbb{Q}$. Let $\overline{X}$ be the closure of X in $\mathbb{P}$. Then $\overline{X}$ is proper iff $\mathcal{F} \supset \mathcal{A}$.*

Proof. Suppose $\overline{X}$ is proper but $\mathcal{A}$ is not contained in $\mathcal{F}$. Consider any equivariant compactification $\mathbb{P} \subset \mathbb{P}'$. Then there exists a cone $C \subset \mathcal{F}'$ with the following property. Its interior C^0 does not intersect $\mathcal{F}$ and contains a point of $\mathcal{A}$. Since $\overline{X}$ is proper, we have $\overline{X} = \overline{X}'$, and therefore $\overline{X}$ does not intersect a T-orbit in $\mathbb{P}' \setminus \mathbb{P}$ that corresponds to C. This contradicts Lemma 2.2.

Suppose $\overline{X}$ is not proper but $\mathcal{A}$ is contained in $\mathcal{F}$. Consider any equivariant compactification $\mathbb{P} \subset \mathbb{P}'$. Since $\overline{X}'$ is proper, it intersects a T-orbit in $\mathbb{P}' \setminus \mathbb{P}$. But $\mathcal{A}$ does not intersect the interior of the corresponding cone of $\mathcal{F}'$ which again contradicts Lemma 2.2. □

Definition 2.4. We say that a pair $(X, \mathbb{P})$ is tropical if the closure $\overline{X}$ of X in $\mathbb{P}$ is a tropical compactification of X, i.e. the multiplication map $\Psi\colon\ T \times \overline{X} \to \mathbb{P}$, $(t, x) \mapsto tx$ is faithfully flat and $\overline{X}$ is proper.

PROPOSITION 2.5. *Assume that* $(X, \mathbb{P})$ *is tropical. Then the fan of* $\mathbb{P}$ *is supported on* $\mathcal{A}$. *Assume, moreover, that* $\mathbb{P}' \to \mathbb{P}$ *is any proper toric morphism. Then* $\overline{X}' \subset \mathbb{P}'$ *is also tropical,* $\overline{X}'$ *is the inverse image of* $\overline{X}$, *and* Ψ' *is the pull back of* Ψ.

Proof. We start with the second statement. Assume $(X, \mathbb{P})$ is tropical and let $f\colon\ \mathbb{P}' \to \mathbb{P}$ be any proper morphism. We claim that $(X, \mathbb{P}')$ is tropical. The only difficulty is to show that the closure of X in $\mathbb{P}'$ is the inverse image of the closure of X in $\mathbb{P}$, the rest is formal. Since the pull-back of a faithfully flat map is faithfully flat, it suffices to show that

$$\Psi' = f^*\Psi,$$

i.e. that $(\overline{X} \times T) \times_{\mathbb{P}} \mathbb{P}'$ is integral. Since $(X \times T) \times_{\mathbb{P}} \mathbb{P}' = X \times T$, this follows from Lemma 2.6 below, where $(\overline{X} \times T) \times_{\mathbb{P}} \mathbb{P}'$ is playing the role of Q, $\mathbb{P}'$ the role of P, and $T \subset \mathbb{P}'$ the role of U. A Cartesian diagram with vertical arrows induced by f

$$\begin{array}{ccccc}
\overline{X}' & \xrightarrow{(Id,e)} & \overline{X}' \times T & \xrightarrow{\Psi' = f^*\Psi} & \mathbb{P}' \\
\downarrow & & \downarrow & & \downarrow \\
\overline{X} & \xrightarrow{(Id,e)} & \overline{X} \times T & \xrightarrow{\Psi} & \mathbb{P}
\end{array}$$

implies that $\overline{X}'$ is an inverse image of $\overline{X}$.

Finally, suppose that $(X, \mathbb{P})$ is tropical but $\mathcal{F}$ is not supported on $\mathcal{A}$. Let $\mathcal{F}'$ be a refinement of $\mathcal{F}$ such that $\mathcal{F}'$ contains a cone C without points of $\mathcal{A}$ in its interior. Then $(X, \mathbb{P}')$ is tropical by the first half of the proof. In particular, Ψ' is surjective. However, by Lemma 2.2, $\overline{X}'$ (and therefore the image of the multiplication map) does not intersect a T-orbit in $\mathbb{P}'$ corresponding to C. □

LEMMA 2.6. *Let* $\phi\colon\ Q \to P$ *be a flat map of schemes with* P *integral and let* U *be an open subset of* P *such that* $\phi^{-1}(U)$ *is integral. Then* Q *is integral.*

Proof. It suffices to prove the following local statement. Suppose $A \subset B$ is a flat extension of rings, A is a domain, $f \in A$, and the localization B_f is a domain. Then B is a domain. The proof goes as follows: since A is a domain, $A \subset A_f$. Since B is flat over A, $B \subset (A_f) \otimes_A B = B_f$. Therefore, B is a domain. □

Remark 2.7. We see that a tropical compactification is defined by a sufficiently fine polyhedral structure on $\mathcal{A}$. It would be interesting to refine Proposition 2.5, for example is it true that $\overline{X}$ is tropical if and only if $\mathcal{F}$ is supported on $\mathcal{A}$? Is there a canonical polyhedral structure on $\mathcal{A}$? Of course, we have not yet proved that tropical compactifications even exist. This will be proved later in this section.

Proof of Theorem 1.4. Recall that a subvariety X of a torus T is called *schön* if it has a tropical compactification with a smooth multiplication map. We claim that if X is schön then any of its tropical compactifications $\overline{X} \subset \mathbb{P}$ has a smooth multiplication map, is regularly embedded (i.e. is locally defined by a regular sequence), normal, has toroidal singularities, and its log canonical line bundle is globally generated and is equal to the determinant of the normal bundle of $\overline{X}$. Moreover, if $\mathbb{P}' \to \mathbb{P}$ is a proper toric morphism then the corresponding map $\overline{X}' \to \overline{X}$ is log crepant.

Assume that X is schön and $(X, \mathbb{P}_0)$ has a smooth multiplication map. Let $(X, \mathbb{P})$ be another tropical object. We can find a tropical object $(X, \mathbb{P}')$ with morphisms $\mathbb{P}' \to \mathbb{P}$, $\mathbb{P}' \to \mathbb{P}_0$. Then Ψ' is the pullback of Ψ and the pullback of Ψ_0 by Proposition 2.5. Therefore all these maps have the same set of geometric fibers. But a flat map is smooth if and only if all its geometric fibers are smooth. Thus Ψ is smooth too.

From the commutative diagram with smooth vertical arrows

$$\begin{array}{ccc} \overline{X} \times T & \xrightarrow{(\Psi, \mathrm{pr}_2)} & \mathbb{P} \times T \\ {\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathbb{P} & = & \mathbb{P} \end{array}$$

it follows that $(\Psi, \mathrm{pr}_2)\colon \overline{X} \times T \to \mathbb{P} \times T$ is a regular embedding by [EGA, 17.12.1]. Since the map

$$\mathbb{P} \times T \to \mathbb{P} \times T, \quad (p, t) \mapsto (t^{-1}p, t)$$

is an automorphism, the obvious embedding $\overline{X} \times T \subset \mathbb{P} \times T$ is also regular. This implies $\overline{X} \subset \mathbb{P}$ is a regular embedding.

Since Ψ is smooth, $\overline{X} \times T$ is normal and has toroidal singularities. Therefore, $\overline{X}$ has these properties as well.

Since $\overline{X} \subset \mathbb{P}$ is regularly embedded, it has a normal bundle $\mathcal{N}$. Since the log canonical bundle of a toric variety is trivial, $\det \mathcal{N}$ is the log canonical bundle of $\overline{X}$ by adjunction. Moreover, for any toric morphism $\mathbb{P}' \to \mathbb{P}$, the $\mathcal{N}'$ is the pullback of $\mathcal{N}$ because $\overline{X}'$ is the inverse image of $\overline{X}$ by Proposition 2.5. Therefore, $\det(\mathcal{N}')$ is the pullback of $\det\mathcal{N}$, i.e. $\overline{X}' \to \overline{X}$ is log crepant.

Finally, dualizing the canonical sequence

$$0 \to \mathcal{N}^\vee \to \Omega^1_{\mathbb{P}}(\log) \to \Omega^1_{\overline{X}}(\log) \to 0$$

of log tangent bundles, we see that $\mathcal{N}$ is globally generated because the log tangent bundle of a toric variety is trivial. Therefore, $\det\mathcal{N}$ is also globally generated. □

Now we prove that tropical compactifications exist by proving that the generalized visible contour compactification $\overline{X}_{vc}$ is tropical. We follow the notation of Definition 1.6 and Theorem 1.7.

Proof of Theorem 1.7. We temporarily denote $\{[Y] \in \mathbb{P}_{Gr} \mid e \in Y\}$ by $\mathcal{K}$. Then it is clear that $\overline{X}_{vc}$ is contained in and is an irreducible component of $\mathcal{K}$: indeed, it suffices to show that $x^{-1} \cdot \overline{X}$ contains e for any $x \in X$. But this is clear: $e = x^{-1} \cdot x$.

Therefore, by Lemma 2.6, it suffices to check that $\mathcal{K} \times T \to \mathbb{P}_{vc}$ is flat. Let $U \subset \mathbb{P}_{vc} \times T$ be the pullback of the universal family in $\mathrm{Hilb}(\mathbb{P}) \times \mathbb{P}$. The morphism $U \to \mathbb{P}_{vc}$ is then flat. But there exists a natural map

$$\mathcal{K} \times T \to U, \quad ([Y], t) \mapsto (t \cdot [Y], t),$$

and this is an isomorphism with inverse $([Y], t) \mapsto (t^{-1}[Y], t)$. In particular the natural morphism $\mathcal{K} \times T \to \mathbb{P}_{vc}$ is flat and isomorphic to $U \to \mathbb{P}_{vc}$. □

Remark 2.8. Laurent Lafforgue suggested that there should be a not constructive proof of the existence of a tropical compactification using an appropriate generalization of Raynauld's flattening stratification theorem over a toric stack $\mathbb{P}/T$.

Now we prove Theorem 1.2 as an immediate corollary of previous results.

Proof of Theorem 1.2. Let $\overline{X}_{vc} \subset \mathbb{P}_{vc}$ be a visible contour compactification. It is tropical by Theorem 1.7. By Proposition 2.5, $\overline{X}'$ is then tropical for any proper morphism $\mathbb{P}' \to \mathbb{P}_{vc}$. We can assume therefore that $(X, \mathbb{P})$ is tropical and $\mathbb{P}$ is smooth by taking a toric resolution of singularities of $\mathbb{P}_{vc}$. Since a boundary of a smooth toric variety has normal crossings, $\overline{X}$ has "combinatorial normal crossings" by dimension reasons (a multiplication map is faithfully flat and therefore equidimensional). Finally, assume that $r = \dim X$ and $p \in \bigcap B_i$. The T-orbit of $\mathbb{P}$ containing p is defined by a regular sequence $f_1, \ldots, f_r$ (being a closed orbit of the smooth toric variety). Since the multiplication map is flat, the pull back of this sequence is a regular sequence $g_1, \ldots, g_r$ on $\overline{X} \times T$. It is immediate from definitions that this sequence cuts out $\{p\} \times T \subset \overline{X} \times T$ set-theoretically. Let

$x_1, \ldots, x_n$ be coordinates on T and let $t = (\lambda_1, \ldots, \lambda_n) \in T$ be a point such that (p, t) does not belong to the union of embedded components of the scheme defined by the ideal $(g_1, \ldots, g_r)$. Then the sequence $(g_1, \ldots, g_r, x_1 - \lambda_1, \ldots, x_n - \lambda_n)$ is regular on $X \times T$. Passing to the localization and changing the order in the regular sequence we see that $(g_1(t), \ldots, g_n(t))$ is a regular sequence in the maximal ideal of p on $\overline{X}$. □

Definition 2.9. Let $\mathbb{P}$ be an auxiliary compactification of T. In the notation of Theorem 1.7, the complete fan $\mathcal{F}_{Gr}$ of $\mathbb{P}_{Gr}$ is called the *Gröbner fan* of $(X, \mathbb{P})$. The *tropical fan* $\mathcal{F}_{vc}$ of $\mathbb{P}_{vc}$ is a subfan of $\mathcal{F}_{Gr}$ of dimension $\dim X$. It follows from Proposition 2.5 and Theorem 1.7 that $\mathcal{F}_{vc}$ is supported on $\mathcal{A}$.

If an auxiliary toric variety $\mathbb{P}$ is a projective space, $\mathcal{F}_{vc}$ is a *tropical variety* of Sturmfels and Speyer [SS], though not quite, as we explain below.

2.10. **Initial Ideals.** [SS] If $\mathbb{P}$ is a projective space then there is an alternative description of a tropical fan provided by initial ideals. Let $\pi\colon V \setminus \{0\} \to \mathbb{P}$ be the canonical projection. It is convenient to temporarily change the notation related to tori. The torus T will be denoted by T' and T will denote $\pi^{-1}(T') \subset V$. Then $T' = T/k^*$ and we can consider all introduced T'-toric varieties as T-varieties and use the notation Λ, $\mathcal{F}_{trop}$, $\mathcal{F}_{Gr}$, ... for T-objects. Let $I \subset k[V]$ be the homogeneous ideal of $\overline{X}$. Let $\Lambda^\vee_+ \subset \Lambda^\vee$ be the standard octant (the fan of a toric variety V).

Any $\chi \in \Lambda^\vee_+$ gives rise to a grading

$$k[V] = \bigoplus_{n \in \mathbb{Z}} k[V]_n, \quad k[V]_n = \{f \in k[V] \mid \chi(z) \cdot f = z^n f\}.$$

For any $f \in k[V]$, we define the initial term $in_\chi(f)$ as the homogeneous component of f of the minimal possible degree. We define the *initial ideal*

$$I_\chi = \{in(f) \mid f \in I\}.$$

Then it is well-known that I_χ describes the flat degeneration, i.e., I_χ is a homogeneous ideal of $\overline{X}_\chi$, where

$$[\overline{X}_\chi] = \lim_{z \to 0} \chi(z)[\overline{X}]$$

is the flat limit. In particular, we can describe the Gröbner fan as follows: $\chi', \chi'' \in \Lambda^\vee_+$ belong to the interior of the same cone of $\mathcal{F}_{Gr}$ if and only if $I_{\chi'} = I_{\chi''}$. A fan $\tilde{\mathcal{F}}_{Gr}$ defined this way is actually a refinement of $\mathcal{F}_{Gr}$ because different initial ideals can produce the same limit (due to the presence of embedded components supported on $\{0\} \subset V$). The corresponding refinement $\mathcal{F}_{trop}$ of $\mathcal{F}_{vc}$ is then a subfan of $\tilde{\mathcal{F}}_{Gr}$ described as follows: $\chi \in \mathcal{F}_{trop}$ if and only if I_χ contains no monomials [SS].

3. Very affine varieties. It is easy to see that a connected affine variety X is a (closed) subvariety of a torus if and only if the ring of regular functions $\mathcal{O}(X)$ is generated by units $\mathcal{O}^*(X)$. In this case we call X *very affine* following a suggestion of Kapranov.

It follows from the definition that any very affine variety X is a subvariety of an *intrinsic torus* defined as $T = \operatorname{Hom}(\Lambda, k^*)$, where $\Lambda = \mathcal{O}^*(X)/k^*$ (it is well-known that it is a finitely generated free $\mathbb{Z}$-module). The embedding $X \subset T$ is unique up to a translation by an element of T. In our main examples (complements to hyperplane arrangements and open cells of Grassmannians) that we study in subsequent sections, the tori T are, in fact, intrinsic.

Any morphism $X \to X'$ of very affine varieties induces the homomorphism $\mathcal{O}^*(X') \to \mathcal{O}^*(X)$ by pull-back and therefore induces the homomorphism of intrinsic tori $T \to T'$. It also obviously induces the map of amoebas $\mathcal{A} \to \mathcal{A}'$.

PROPOSITION 3.1. *If $X \to X'$ is a dominant morphism of very affine varieties (e.g., an open immersion) then the corresponding map of amoebas is surjective.*

Proof. We consider X and X' as closed subvarieties of their intrinsic tori T and T'. Let $f\colon X \to X'$ be a dominant morphism. We denote the induced map of amoebas by the same letter $f\colon \mathcal{A} \to \mathcal{A}'$. Let Z be a connected Weyl divisor of X' that contains $X' \setminus f(X)$. We consider Z as a subvariety of the torus T', let $\mathcal{B}$ be the corresponding amoeba in $(\Lambda'_{\mathbb{Q}})^\vee$. The crucial observation [EKL] is that $\mathcal{B}$ (or more precisely any fan supported on it) is equidimensional of dimension equal to the Krull dimension of Z while the dimension of $\mathcal{A}'$ is equal to $\dim X'$.

It follows that for any $p \in \mathcal{A}'$ there exists $x' \in (X' \setminus Z)(\overline{K})$ such that $p = \deg x'$. It is clear that there exists $x \in X(\overline{K})$ such that $f(x) = x'$. Therefore, $p \in f(\mathcal{A})$. □

Remark 3.2. Using the same argument, one can prove a slightly stronger result: it is not necessary to assume that tori T and T' are intrinsic. It suffices to assume that the pull-back of units induces a homomorphism $T \to T'$. This assumption is also necessary: without it, there is even no natural map of amoebas $\mathcal{A} \to \mathcal{A}'$.

Remark 3.3. Proposition 3.1 can be used to construct tropical compactifications of X': find a “simple” variety X that dominates X', find an amoeba $\mathcal{A}$ of X, find an amoeba $\mathcal{A}'$ of X' as the image of $\mathcal{A}$. Then find a fan supported on $\mathcal{A}'$ and try to prove (e.g. using local calculations in toric charts) that the corresponding compactification is tropical. Applications of this algorithm to the study of moduli of Del Pezzo surfaces will be published elsewhere.

Remark 3.4. It follows (using Proposition 2.5) that for any tropical compactification $\overline{X}'$, there exists a tropical compactification $\overline{X}$ and a morphism $\overline{X} \to \overline{X}'$ that extends $X \to X'$. This motivates the following conjectures: any algebraic variety X contains a schön very affine open subset U (resp. any algebraic variety X is dominated by a schön very affine variety U). This conjecture is obviously related to the resolution of singularities (resp. alteration of singularities as in [J]).

4. Complements of hyperplane arrangements. Consider the complement of a hyperplane arrangement

$$X = \mathbb{P}^r \setminus \{H_1, \ldots, H_n\}. \qquad (\circ)$$

We assume that the arrangement is essential and connected.

$\mathbb{P}^r$ admits an embedding in $\mathbb{P}^{n-1}$ such that the hyperplanes H_i are intersections with coordinate hyperplanes of $\mathbb{P}^{n-1}$. It follows that X is a closed subvariety of a torus $T = (k^*)^{n-1}$. It is easy to see that this torus is intrinsic.

We take $\mathbb{P}^{n-1}$ as the auxiliary compactification of the visible contour construction of Definition 1.6. Connectedness of the arrangement implies $T_X = \{e\}$. The component of the Hilbert scheme of $\mathbb{P}^{n-1}$ containing $\overline{X} = \mathbb{P}^r$ is obviously the Grassmannian $G(r+1,n)$. Let $[\overline{X}] \in G(r+1,n)$ be the corresponding point. The toric Gröbner variety $\mathbb{P}_{Gr}$ is the normalization of a closure of $T\cdot[\overline{X}] \subset G(r+1,n)$.

It follows that $\overline{X}_{vc}$ is equal to the closure of $X^{-1}\cdot[\overline{X}]$ in $\mathbb{P}_{Gr}$. In this strange looking formula X^{-1} means inverses of elements of $X \subset T$. This is precisely Kapranov's visible contour (he takes the closure in $G(r+1,n)$ rather than $\mathbb{P}_{Gr}$ but this does not matter: $\overline{X}_{vc} \subset \mathbb{P}_{vc}$ and $\mathbb{P}_{vc}$ is isomorphic to its image in $G(r+1,n)$ (i.e. this image is normal) by a result of White [W]).

We continue with the proof of Theorem 1.5.

Let $\Psi\colon\ T \times \overline{X}_{vc} \to \mathbb{P}_{vc}$ be the multiplication map. Since the universal family of the Grassmannian is smooth (being a projective bundle), we see as in the proof of Theorem 1.7 that Ψ is not just flat but smooth, i.e. X is schön. By Theorem 1.4, $\overline{X}_{vc}$ is normal and has toroidal singularities. Let $G_e \subset G(r+1,n)$ be the subvariety of subspaces containing $e = (1,\ldots,1) \in T$. Obviously, G_e is isomorphic to $G(r,n-1)$ and is regularly embedded in $G(r+1,n)$ with normal bundle being the universal quotient bundle of G_e. By Theorem 1.7, $\overline{X}_{vc}$ is a scheme-theoretic intersection of $\mathbb{P}_{vc}$ with G_e. In particular, the normal bundle of $\overline{X}_{vc} \subset \mathbb{P}_{vc}$ in this case is the pullback of the universal quotient bundle of G_e. By Theorem 1.4, the log canonical bundle of $\overline{X}_{vc}$ is the determinant of its normal bundle. It follows that it is very ample being the restriction of a Plücker polarization of G_e. This finishes the proof of Theorem 1.5. □

Example 4.1. [CQ2] Assume that $X = \mathbb{P}^2 \setminus \{L_1,\ldots,L_n\}$. Let the *multiplicity* of $p \in \mathbb{P}^2$ be the number of lines passing through it. Let $\overline{X}_{wond}$ be the blow up of $\mathbb{P}^2$ in the set of points of multiplicity at least 3. Restricting the log canonical line bundle on (proper transforms of) lines and exceptional divisors, we see that it is ample, and therefore $\overline{X}_{wond} = \overline{X}_{vc}$ with the following exception. Suppose the configuration of lines contains a line L and two points $a, b \in L$ such that any other line passes through a or b. In this case the restriction of the log canonical line bundle on L is trivial, and therefore $\overline{X}_{vc} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (the blow-down of the proper transform of L in $\overline{X}_{wond} = \mathrm{Bl}_{a,b}\,\mathbb{P}^2$).

This example generalizes to the higher dimensions as follows. Since $\overline{X}_{vc}$ is the log canonical model of X, it is natural to wonder how $\overline{X}_{vc}$ is related to compactifications of X with a normally crossing boundary. They can be constructed as subsequent blow ups of $\mathbb{P}^r$ along (proper transforms of) projective subspaces required to make the proper transform of the hyperplane arrangement a divisor with normal crossings. For example, one can blow up all possible partial intersections of hyperplanes and obtain the variety BL of [CQ2, §5]. The most economical blow up is the so-called *wonderful compactification* $\overline{X}_{wond}$ constructed in [CP]. In fact, there is a natural poset of compactifications with normal crossings constructed in [FY] such that BL is the maximal and $\overline{X}_{wond}$ is the minimal element of this poset. These compactifications $\overline{X}_{nest}$ are indexed by the so called nesting sets. Moreover, any $\overline{X}_{nest}$ is the closure of X in a certain smooth toric variety $\mathbb{P}_{nest}$. It is shown in [FS] by a combinatorial argument that the fan $\mathcal{F}_{nest}$ refines $\mathcal{F}_{vc}$ and in particular $\overline{X}_{nest}$ is tropical by Proposition 2.5 and there is a canonical map $\overline{X}_{nest} \to \overline{X}_{vc}$. This map is log crepant by Theorem 1.4, i.e. the logcanonical divisor $K+B$ is globally generated on $\overline{X}_{nest}$ and gives a regular map $\overline{X}_{nest} \to \overline{X}_{vc}$ identical on X. Example 4.1 shows that $\overline{X}_{wond}$ is not necessarily equal to $\overline{X}_{vc}$. The combinatorial condition to guarantee $\overline{X}_{wond} = \overline{X}_{vc}$ was found in [FS] (geometrically it means that any strata of $\overline{X}_{wond}$ is of log general type).

5. Tropical quotients of Grassmannians.

5.1. **Tropical Compactifications as Compact Quotient Spaces.** Let X be a closed subvariety of a torus T. Here we will be interested in the case $T_X \neq \{e\}$. Since $\overline{X}_{vc}$ compactifies X/T_X, it is natural to compare $\overline{X}_{vc}$ with other known compact quotients. The most closely related ones are Hilbert/Chow quotients, see [KSZ], [CQ1]. The construction goes as follows. We first take any auxilliary compact toric variety $\mathbb{P}$ of T as in the construction of $\overline{X}_{vc}$. Let $Z \subset \overline{X}$ be the closure of a generic T_X-orbit in $\overline{X}$. Then we have an embedding

$$T/T_X \hookrightarrow \mathrm{Hilb}\,(\mathbb{P}), \quad t \mapsto t \cdot [Z],$$

where we take a component of a Hilbert scheme containing $[Z]$ (note that the construction of $\overline{X}_{vc}$ uses a completely different component). Let $\mathbb{P}_{Hilb}$ be the normalization of the closure of T/T_X in $\mathrm{Hilb}\,(\mathbb{P})$. The closure of X/T_X in $\mathbb{P}_{Hilb}$ is the *Hilbert quotient* $\overline{X}_{Hilb}$ (one can also use a Chow variety of $\mathbb{P}$ or its multigraded Hilbert scheme to obtain closely related quotients).

The difference between $\overline{X}_{Hilb}$ and $\overline{X}_{vc}$ can be summarized as follows: $\overline{X}_{vc}$ parametrizes T-translates of $\overline{X}$ (and their limit degenerations) that contain Z, while $\overline{X}_{Hilb}$ parametrizes T-translates of Z (and their limits – broken toric varieties) in $\mathbb{P}$ that are contained in $\overline{X}$. More precisely, let S be the set of these generic orbits and their limits. S has a natural scheme structure. In general S is reducible (example is given in [CQ2]) and $\overline{X}_{Hilb}$ is the reduction of the principal component of S. The reason why $\overline{X}_{vc}$ is better than $\overline{X}_{Hilb}$ is because a containment-scheme (subfamily of a family of subschemes that contain a fixed subvariety) in general behave better than an inclusion-scheme (subfamily of a family of subschemes contained in a fixed subvariety).

Definition 5.2. Let $\mathbb{P}_{Laf} \subset \mathbb{P}_{Hilb}$ be the open chart (introduced by Lafforgue in the case of the Grassmannian) consisting of T-orbits that intersect $\overline{X}_{Hilb}$. Let $\mathcal{F}_{Laf}$ be the corresponding fan.

$\overline{X}_{Hilb}$ is often not tropical. Here we give a stupid example with $T_X = e$; an example with $T_X \neq \{e\}$ is given in Remark 5.3. Let X be a complement to a not normally crossing connected hyperplane arrangement ($\circ$). Obviously

$$\overline{X}_{Hilb} = \mathbb{P}^r \subset \mathbb{P}_{Hilb} = \mathbb{P}^{n-1}$$

because $T_X = \{e\}$. Since the boundary does not have combinatorial normal crossings, the multiplication map is not flat by Theorem 1.2. On the other hand, $\overline{X}_{vc}$ magically incorporates all necessary blow ups. The special configuration of the Example 4.1 also shows that in general there is no regular map $\overline{X}_{vc} \to \overline{X}_{Hilb}$.

In the remainder of this section we study the quotient of a Grassmannian by a maximal torus. Let $X = G^0(r,n)$ be the open cell in the Grassmannian $G(r,n)$ given by non-vanishing of all Plücker coordinates. Consider the Plücker embedding

$$X = G(r,n) \subset \mathbb{P} = \mathbb{P}(\Lambda^r k^n).$$

The complement to coordinate hyperplanes in $\mathbb{P}$ is a torus T that contains X as a closed subvariety. It is easy to see that T is an intrinsic torus of X. $T_X = (k^*)^n/k^*$ is the diagonal torus. We fix $\mathbb{P}$ as an auxiliary toric variety necessary to construct the Hilbert quotient $\overline{X}_{Hilb}$ and the generalized visible contour compactification $\overline{X}_{vc}$ and compare them. In fact, instead of $\overline{X}_{vc}$, we will consider its slight modification $\overline{X}_{trop}$ defined in 2.10 using initial ideals. I don't know if $\overline{X}_{vc} = \overline{X}_{trop}$. The fan $\mathcal{F}_{trop}$ is the *tropical Grassmannian* of Sturmfels and Speyer [SS]. $\overline{X}_{Hilb}$ is Kapranov's Chow quotient of Grassmannian introduced in [CQ1].

Remark 5.3. By the Gelfand–Macpherson correspondence X/T_X is identified with the moduli space $X(r,n)$ of arrangements of n hyperplanes in $\mathbb{P}^{r-1}$ in linearly general position. By [HKT], $\overline{X}_{Hilb}$ has a functorial description as the moduli space of stable pairs of log general type (and thus it serves as a template for the study of moduli spaces of varieties of general type). Description of fibers of the universal family over $\overline{X}_{Hilb}$ (higher dimensional analogs of stable rational curves) were found in [Laf], their natural crepant resolutions were found in [CQ2]. However, by [CQ2] the geometry of $\overline{X}_{Hilb}$ is terrible: for any scheme S of finite type over $\operatorname{Spec}\mathbb{Z}$, there is a strata in $\overline{X}_{Hilb}$ isomorphic to the open subset $U \subset S \times \mathbb{A}^r$ such that the projection $U \to S$ is onto. Also, if $n \geq 9$ then $\overline{X}_{Hilb}$ is not a log canonical model of X. Moreover, it is proved in [CQ2] that the fan of $\mathbb{P}_{Hilb}$ for $n \geq 9$ contains cones of dimension bigger than dimension of the amoeba. In particular, $\overline{X}_{Hilb}$ is not tropical for $n \geq 9$.

THEOREM 5.4. *There exists a toric morphism* $\mathbb{P}_{trop} \to \mathbb{P}_{Laf}$.

In particular, there exists a morphism $\overline{X}_{trop} \to \overline{X}_{Hilb}$. In some cases we can say more. The following theorem is a simple compilation of known results:

THEOREM 5.5. *Let* $X = G^0(2,n)$. *Then* $\overline{X}_{Hilb} = \overline{X}_{trop} = \overline{M}_{0,n}$, *the Grothendieck–Knudsen moduli space of stable rational curves.*

Proof. Fans $\mathcal{F}_{Laf}$ and $\mathcal{F}_{vc}$ were calculated in [CQ1] and [SS], respectively. They are the same, the so-called *space of phylogenetic trees*. Therefore

$$\overline{X}_{trop} = \overline{X}_{Hilb} = \overline{M}_{0,n}$$

the last equality is a result of Kapranov [CQ1]. □

Remark 5.6. Notice that $M_{0,n}$ is also the complement of the *braid arrangement*

$$\{x_i = x_j \mid 1 \leq i < j \leq n-1\} \subset \mathbb{P}^{n-3},$$

where $\sum x_i = 0$. In particular, it has another visible contour compactification defined in the framework of Theorem 1.5. Since $\overline{M}_{0,n}$ is the log canonical model of $M_{0,n}$, this latter compactification is also isomorphic to $\overline{M}_{0,n}$ by Theorem 1.5 and gives its Kapranov's blow up model defined in [CQ1].

The next result is joint with Hacking and Keel. Its proof will appear elsewhere. Conjecturally, its analogue holds for $G(3,7)$ and $G(3,8)$.

THEOREM 5.7. *Let* $X = G^0(3,6)$. *Then* X *is schön,* $\overline{X}_{Hilb}$ *is its log canonical model. It has* 40 *isolated singularities.* $\overline{X}_{trop}$ *is a crepant resolution of* $\overline{X}_{Hilb}$.

The rest of the paper is occupied by the proof of Theorem 5.4.

Proof. Let $\Delta = \Delta(r,n) \subset \mathbb{R}^n$ be the *hypersymplex*, i.e. the convex hull of points

$$e_I := \sum_{i\in I} e_i, \quad \text{where} \quad I \subset N := \{1,\dots,n\}, \quad |I| = r.$$

It is well known that all these points are vertices of Δ and we frequently identify Δ with its set of vertices if it does not cause confusion.

Definition 5.8. Let

$$\mathcal{H} = \{H_1, \ldots, H_n\} \subset \mathbb{P}^{r-1}$$

be an essential hyperplane arrangement (not all hyperplanes pass through a point). In contrast to the Section 4, we allow some hyperplanes to appear more than once (to be multiple). Then the *matroid polytope* $P_{\mathcal{H}} \subset \Delta(r,n)$ of $\mathcal{H}$ is the convex hull of vertices e_I such that $H_{i_1}, \ldots, H_{i_r}$ are linearly independent.

Remark 5.9. So for example Δ itself is a matroid polytope of a configuration of hyperplanes in linearly general position. $P_{\mathcal{H}}$ has the maximal dimension if and only if $\mathcal{H}$ is a connected configuration, i.e. has no projective automorphisms.

Definition 5.10. For any $i_0 \in N$, we define a *restricted configuration* $R_{i_0}(\mathcal{H}) \subset H_{i_0}$ as the set $H_i \cap H_{i_0}$ for any i such that $H_i \neq H_{i_0}$ (so the indexing set of $R_{i_0}(\mathcal{H})$ can be strictly smaller than $N \setminus \{i_0\}$). For any subset $K \subset N$, we define a *deleted configuration* $C_K(\mathcal{H}) := \mathcal{H} \setminus \{H_k\}_{k \in K} \subset \mathbb{P}^{r-1}$. We say that $\mathcal{H}$ is *more constrained* than $\mathcal{H}'$ if any linearly independent subset of $\mathcal{H}$ is linearly independent in $\mathcal{H}'$ but not vice versa (this is equivalent to the strict inclusion $P_{\mathcal{H}} \subset P_{\mathcal{H}'}$). More generally, if $\mathcal{H}$ is indexed by M, $\mathcal{H}'$ is indexed by N, and $M \subset N$ then we say that $\mathcal{H}$ is more constrained than $\mathcal{H}'$ if $\mathcal{H}$ is more constrained than $C_{N \setminus M}(\mathcal{H}')$.

LEMMA 5.11. *Let $r > 2$. Let $\mathcal{H}, \mathcal{H}' \subset \mathbb{P}^{r-1}$ be connected configurations of n hyperplanes such that $\mathcal{H}$ is more constrained than $\mathcal{H}'$. Then there exists $i_0 \in N$ such that $R_{i_0}(\mathcal{H})$ is connected and more constrained than $R_{i_0}(\mathcal{H}')$ (which in this case is of course also automatically connected).*

Let $\mathcal{H}, \mathcal{H}' \subset \mathbb{P}^1$ be connected configurations of $n > 4$ points such that $\mathcal{H}$ is more constrained than $\mathcal{H}'$. Then there exists an index $i_0 \subset N$ such that $C_{i_0}(\mathcal{H})$ is connected and more constrained than $C_{i_0}(\mathcal{H}')$ (which in this case is also connected).

Proof. Let $\{H_1, \ldots, H_k\} \subset \mathcal{H}$ be a minimal set of linearly dependent hyperplanes such that the corresponding hyperplanes in $\mathcal{H}'$ are linearly independent. Then we can find linear equations of these hyperplanes such that $f_1, \ldots, f_{k-1}$ are linearly independent and $f_k = f_1 + \cdots + f_{k-1}$. Let $\hat{H}_k \in \mathcal{H}$ be a hyperplane such that $\{H_1, \ldots, H_{k-1}, \hat{H}_k\}$ is linearly independent. If $k = 2$ then we assume, in addition, that $H'_1, H'_2, \hat{H}'_2$ is linearly independent (here we use $r > 2$). Finally, we construct a basis $\{H_1, \ldots, H_{k-1}, \hat{H}_k, H_{k+1}, \ldots, H_r\} \subset \mathcal{H}$. We define a graph Γ with vertices $\{1, \ldots, r\}$ as follows: i and j are connected by an edge if and only if there exists a hyperplane $H \in \mathcal{H}$ such that in the expression of its equation in our basis i-th and j-th coefficients are both nontrivial. Then it is immediate from the calculation of a (trivial) automorphism group of a configuration in our basis that this graph is connected. Therefore, it has at least two vertices that can be dropped without ruining the connectedness. We take i_0 to be any of these vertices unless $k = 2$ in which case we take i_0 to be any of these vertices not equal to 1. Now all claims follow by immediate inspection.

The statement about $\mathbb{P}^1$ is obvious. □

5.12. Pictures. Let $R \subset K$ be the DVR of power series. Let

$$\mathcal{H}(z) = \{H_1(z), \ldots, H_n(z)\} \subset \mathbb{P}^{r-1}(K)$$

be the collection of (one-parameter families) of hyperplanes in linearly general position. Any choice of a K-frame F (the projectivization of a basis in K^r) gives a canonical way to extend this K-point of $\mathbb{P}^{r-1}$ to the R-point. Notice that frames related by an element of $PGL_r(R)$ will give isomorphic families over $\operatorname{Spec} R$, i.e. points of the homogeneous space $PGL_r(K)/PGL_r(R)$ (the so called *affine building*) give a family defined upto an isomorphism. The special fiber of this family is $\mathbb{P}^{r-1}$ with a limiting configuration of hyperplanes $\mathcal{H}^F(0) = \{H_1^F(0), \ldots, H_n^F(0)\} \subset \mathbb{P}^{r-1}$ that depends on the choice of a frame. Most of the time, this configuration is not even essential. The set of frames for which this configuration is essential is, by definition, a *membrane* [CQ2]. It is proved there that matroid polytopes of these essential configurations give a *matroid decomposition*, i.e. a paving of Δ by a finite number of matroid polytopes. As mentioned above, cells (polytopes of the maximal dimension) of this decomposition correspond to limiting configurations that are connected.

Remark 5.13. A membrane is a polyhedral complex surprisingly homeomorphic to the nonarchimedean amoeba of a very affine variety $\mathbb{P}^{r-1}(K)\setminus\mathcal{H}(z)$ defined over K [CQ2]. In [DT], the analogue of a membrane (denoted there by $T_{E,v}$) was introduced for any, not necessarily realizable, valuated matroid (I am grateful to the referee for this reference).

Definition 5.14. We say that a configuration of hyperplanes $\mathcal{H}(z)$ over K in linearly general position is *more constrained* than $\mathcal{H}'(z)$ if the matroid decomposition induced by $\mathcal{H}'(z)$ is coarser than the matroid decomposition induced by $\mathcal{H}(z)$.

LEMMA 5.15. *Let $r > 2$. Let $\mathcal{H}(z), \mathcal{H}'(z) \subset \mathbb{P}^{r-1}(K)$ be configurations of n hyperplanes in linearly general position such that $\mathcal{H}(z)$ is more constrained than $\mathcal{H}'(z)$. Then there exists $i_0 \in N$ such that $R_{i_0}(\mathcal{H}(z))$ is more constrained than $R_{i_0}(\mathcal{H}'(z))$ (also notice that $R_{i_0}(\mathcal{H}(z))$ and $R_{i_0}(\mathcal{H}'(z))$ are automatically in linearly general position because $\mathcal{H}(z)$ and $\mathcal{H}'(z)$ are).*

Let $\mathcal{H}(z), \mathcal{H}'(z) \subset \mathbb{P}^1(K)$ be configurations of different $n > 4$ points such that $\mathcal{H}(z)$ is more constrained than $\mathcal{H}'(z)$. Then there exists an index $i_0 \subset N$ such that $C_{i_0}(\mathcal{H})$ is more constrained than $C_{i_0}(\mathcal{H}')$ (also notice that $C_{i_0}(\mathcal{H}(z))$ and $C_{i_0}(\mathcal{H}'(z))$ are automatically in linearly general position because $\mathcal{H}(z)$ and $\mathcal{H}'(z)$ are).

Proof. There are frames F and F' for $\mathcal{H}(z)$ and $\mathcal{H}'(z)$ that produce matroid polytopes of maximal dimension embedded in each other. This means that the corresponding limiting configuration $\mathcal{H}^F(0)$ is more constrained then $\mathcal{H}^{F'}(0)$. Now apply Lemma 5.11. ☐

Let $X = G^0(r,n)$. We identify the Plücker vector space V with k^Δ, the intrinsic torus T with $(k^*)^\Delta$ (using the convention about tori from (2.10)), and the lattice $\Lambda^\vee$ with $\mathbb{Z}^\Delta$. Let $\mathcal{I}$ be the ideal of the cone over $G(r,n)$ in k^Δ.

5.16. Description of Fans. As a set, the "tropical Grassmannian" $\mathcal{F}_{trop}$ is described in [SS, Theorem 3.8]. Namely, $\omega \in \mathbb{Z}^\Delta$ belongs to $\mathcal{F}_{trop}$ if and only if there exists $[L] \in X_K$ (or equivalently the one-parameter family of configurations of hyperplanes $\mathcal{H}(z) \subset \mathbb{P}^{r-1}(K)$ as above by the Gelfand–Macpherson transform) such that

$$\omega(I) = \deg \Delta_I(L), \tag{5.16.1}$$

where Δ_I is the Plücker coordinate that corresponds to $I \subset N$, $|I| = r$, and $\deg\colon K \to \mathbb{Z}$ is the standard valuation. The matroid decomposition corresponding to $\mathcal{H}(z)$ is can be recovered from ω as follows. Let $\Delta_\omega \subset \Delta \times \mathbb{Z} \subset \mathbb{R}^{n+1}$ be the convex hull of points $(I, \omega(I))$ for $I \subset N$, $|I| = k$. Then all these points are vertices of Δ_ω and projections of faces of Δ_ω on Δ give a matroid decomposition of Δ. By [Laf], two functions in $\mathbb{Z}^\Delta$ belong to the interior of the same cone in $\mathcal{F}_{Laf}$ if and only if they induce the same matroid decomposition and the inclusion of cones in $\mathcal{F}_{Laf}$ corresponds to the coarsening of the paving.

5.17. Face Maps. We have to remind definitions of face maps and cross-ratios [CQ1], [Laf], [CQ2]. For any $i_0 \in N$, restriction and deletion of configurations of hyperplanes in linearly general position can be extended to morphisms

$$\overline{X}_{Hilb}(r,n) \to \overline{X}_{Hilb}(r-1,n-1), \quad \overline{X}_{Hilb}(r,n) \to \overline{X}_{Hilb}(r-1,n).$$

These morphisms are induced by morphisms of toric varieties related to the natural inclusions

$$\Delta(r-1,n-1) = \{p \in \Delta(r,n) \mid x_{i_0}(p) = 1\}$$

and

$$\Delta(r-1,n) = \{p \in \Delta(r,n) \mid x_{i_0}(p) = 0\}.$$

More precisely, let us interpret $\mathcal{F}_{Laf} \subset \mathbb{Q}^\Delta$ as a set of functions on the hypersymplex. Then restriction and deletion are toric morphisms given by maps of fans obtained by restricting functions to the boundary of the hypersymplex.

In particular, for any collection of hyperplanes

$$\{H_{i_1},\dots,H_{i_4};H_{j_1},\dots,H_{j_{r-2}}\},\quad \text{where}\quad I\cap J=\emptyset,$$

there is a *cross-ratio map*

$$\overline{X}_{Hilb}\to\overline{X}_{Hilb}(2,4)=\overline{M}_{0,4}=\mathbb{P}^1:$$

delete all hyperplanes not in $I\cup J$ and then consecutively restrict on all hyperplanes from J. The cross-ratio map is induced by the toric morphism $\mathbb{P}_{Laf}\to\mathbb{P}^2\setminus\{e_1,e_2,e_3\}$ given by the inclusion of the octahedron $\Delta(2,4)\subset\Delta(r,n)$, where $\Delta(2,4)$ has vertices

$$e_{i_a}+e_{i_b}+e_{j_1}+\cdots+e_{j_{r-2}}\quad\text{for}\quad(a,b)\subset\{1,2,3,4\}.$$

One can predict whether the cross-ratio is equal to 0, 1, or ∞ by looking at the decomposition of the octahedron $\Delta(2,4)$ induced by the matroid decomposition of Δ because $\{0,1,\infty\}\subset\mathbb{P}^1$ is the intersection of $\mathbb{P}^1$ with the divisorial boundary of $\mathbb{P}^2\setminus\{e_1,e_2,e_3\}$. The values of 0, 1, and ∞ correspond to three different ways to decompose the octahedron as a union of two pyramids.

LEMMA 5.18. *Let $\omega\in\mathcal{F}_{trop}$. Then the initial ideal $\mathcal{I}_\omega$ determines uniquely whether any cross-ratio is equal to* 0, 1, ∞, *or none of the above.*

Proof. Indeed, $\mathcal{I}$ includes the simplest Plücker relation

$$f=\Delta_{12J}\Delta_{34J}-\Delta_{13J}\Delta_{24J}+\Delta_{14J}\Delta_{23J}.$$

This is the unique (up to a scalar) element of $\mathcal{I}$ of its $(k^*)^n$ weight (this easily follows by inspection by looking at other Plücker relations). Since $\mathcal{I}_\omega$ does not contain monomials, there are only 4 possibilities for $in_\omega(f)$: either $in_\omega(f)=f$, i.e.,

$$\omega_{12J}+\omega_{34J}=\omega_{13J}+\omega_{24J}=\omega_{14J}+\omega_{23J},$$

in which case $\Delta(2,4)$ is obviously not split. Or $in_\omega(f)$ can be a sum of two monomials, e.g.,

$$\Delta_{12J}\Delta_{34J}-\Delta_{13J}\Delta_{24J},$$

in which case

$$\omega_{12J}+\omega_{34J}=\omega_{13J}+\omega_{24J}<\omega_{14J}+\omega_{23J}$$

and $\Delta(2,4)$ is split into 2 pyramids. In any case we see that $\mathcal{I}_\omega$ determines uniquely whether any cross-ratio is equal to 0, 1, ∞, or none of the above. □

Now we can prove the theorem. We have to show that the interior of any cone in $\mathcal{F}_{trop}$ belongs to the interior of a cone in $\mathcal{F}_{Laf}$. It suffices to show that if $\omega, \omega' \in \mathcal{F}_{trop} \cap \mathbb{Z}^{\Delta}$ and ω' gives a coarser matroid decomposition than ω then $\mathcal{I}_\omega \neq \mathcal{I}_{\omega'}$. Choose $\mathcal{H}(z)$ and $\mathcal{H}'(z)$ that correspond to ω and ω' as in 5.16. Then $\mathcal{H}(z)$ and $\mathcal{H}'(z)$ is more constrained than $\mathcal{H}'(z)$ by definition.

Applying Lemma 5.15 consecutively, we can find subsets I and J with $|I| = 4$ and $|J| = r-2$ such that $R_J C_{(I\cup J)^c}(\mathcal{H}(z))$ is more constrained than $R_J C_{(I\cup J)^c}(\mathcal{H}'(z))$. This means that the octahedron $\Delta(2,4)$ corresponding to I and J is split by ω and not split by ω'. By Lemma 5.18, it follows that $\mathcal{I}_\omega \neq \mathcal{I}_{\omega'}$. □

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS AT AMHERST, AMHERST, MA, 01003-9305

REFERENCES

[BG] R. Bieri and J. R. J. Groves, The geometry of the set of characters induced by valuations, *J. Reine Angew. Math.* **347** (1984), 168–195.

[CP] C. De Concini and C. Procesi, Wonderful models of subspace arrangements, *Sel. Math.* **1** (1995), 459–494.

[DT] A. Dress and W. Terhalle, The tree of life and other affine buildings, *Doc. Math.*, J. DMV, Extra Vol. ICM Berlin 1998, vol. III, pp. 565–574.

[EKL] M. Einsiedler, M. Kapranov, and D. Lind, Non-archimedean amoebas and tropical varieties, preprint, 2004, math.AG/0408311.

[FS] E. Feichtner and B. Sturmfels, Matroid polytopes, nested sets and Bergman fans, preprint, 2004, math.CO/0411260.

[FY] E. Feichtner and S. Yuzvinsky, Chow rings of toric varieties defined by atomic lattices, *Invent. Math.* **155**(3) (2004), 515–536.

[EGA] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique. IV. Étude locale des schémas et des morphismes de schémas, *Inst. Hautes Études Sci. Publ. Math.* **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[HKT] P. Hacking, S. Keel, and J. Tevelev, Compactification of the moduli space of hyperplane arrangements, *J. of Algebraic Geom.* **15** (2006), 657–680.

[J] A. de Jong, Families of curves and alterations, *Ann. Inst. Fourier (Grenoble)* **47** (1997), no. 2, 599–621.

[CQ1] M. Kapranov, Chow Quotients of Grassmannians I, *Advances in Soviet Mathematics* **16** (1993), 29–110.

[KSZ] M. Kapranov, B. Sturmfels, and A. Zelevinsky, Quotients of toric varieties, *Math. Ann.* **290** (1991), no. 4, 643–655.

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model program, *Adv. Stud. Pure Math.* **10** (1987), 283–360.

[CQ2] S. Keel and J. Tevelev, Chow Quotients of Grassmannians II, *Duke Math. J.* **134** no. 2 (2006), 259–311.

[KT] S. Keel and J. Tevelev, Equations for $\bar{M}_{0,n}$, preprint, 2005, math.AG/0507093.

[Laf] L. Lafforgue, *Chirurgie des grassmanniennes*, *CRM Monograph Ser.*, vol. 19, Amer. Math. Soc., Providence, RI, 2003.

[R] P. Ribenboim, Fields: algebraically closed and others, *Manuscripta Math.* **75** (1992), 115–150.

[SS] D. Speyer and B. Sturmfels, The Tropical Grassmannian, *Adv. Geom.* **4** (2004), 389–411.

[V1] A. Varchenko, Newton Polyhedra and Estimations of Oscillatory Integrals, *Funct. Anal. Appl.* **10** (1976), 175–196.

[V2] ———, Zeta-function of monodromy and Newton's diagram, *Invent. Math.* **37** (1976), no. 3, 253–262.

[W] N. White, The basis monomial ring of a matroid, *Adv. Math.* **24** (1997), 292–297.