Problem Solving Seminar. Worksheet 6. Algebraic Techniques.

Move everything to one side. Factor. Expand. Complete the square! $x^n - y^n = ?$, $x^{2m+1} + y^{2m+1} = ?$. A degree n polynomial is determined by its values at $n+1$ points. Coefficients of a polynomial in terms of its roots. How to get sums of powers of roots? Rational roots theorem (if a polynomial with integer coefficients has a rational root then ???). Long division of polynomials.

1. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

2. $1/(x+1)(x+2)(x+3) = ?/(x+1) + ?/(x+2) + ?/(x+3)$.

3. If $P(x)$ is a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, \ldots, n$, determine $P(n+1)$.

4. Show that $\sqrt{2}$ is irrational.

5. Suppose all 2×2 minors of a 3×3 matrix with integral coefficients are divisible by 5. Show that its determinant is divisible by 25.

6. Solve the system of equations

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12 \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24 \\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48 \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96 \end{aligned}$$

7. It is known that a quadratic equation has either 0, 1, or 2 unique real solutions. But consider the equation

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$$

where a, b, and c are distinct. Notice that $x = a$, $x = b$, and $x = c$ are all solutions — how can this equation have three solutions?

8. Show that each number in the sequence 49, 4489, 444889, 44448889, ... is a perfect square.

9. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

10. *(The interpolation formula)* Suppose a_1, ..., a_n are distinct numbers, and b_1, ..., b_n are given numbers, and $P(x)$ is a degree at most $n-1$ polynomial such that $P(a_i) = b_i$ for all i. Show that

$$P(x) = b_1 \frac{(x-a_2)(x-a_3)\cdots(x-a_n)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + b_2 \frac{(x-a_1)(x-a_3)\cdots(x-a_n)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} + \cdots + b_n \frac{(x-a_1)(x-a_2)\cdots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}.$$

11. Prove that $(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}$ is rational.

12. Solve

$$(x^2-3x-4)(x^2-5x+6)(x^2+2x) + 30 = 0.$$