Problem Solving Seminar. Worksheet 4. Number Theory.

Basic number theory concepts: modular arithmetic, unique factorization, greatest common divisor is a linear combination, Chinese Remainder Theorem, positional notation, Fermat's little theorem $a^p \equiv a \pmod{p}$, Euler ϕ-function $\phi(n) = \#$ numbers between 1 and n relatively prime to n, $a^{\phi(n)} \equiv 1 \pmod{n}$ if $\gcd(a, n) = 1$.

1. What are the last two digits of 3^{1234}?
2. Is $\binom{100}{36}$ even or odd?
3. If $2n + 1$ and $3n + 1$ are both perfect squares, show that n is divisible by 40.
4. Prove that if $ad - bc = 1$ then $g.c.d.(a + b, c + d) = 1$.
5. Let A be a 3×3 matrix with integer entries. Prove that if all 2×2 minors of A are divisible by 7 then $\det A$ is divisible by 49. (A *minor* is a determinant of a submatrix obtained by choosing some rows and columns in A.)
6. Prove that there are infinitely many primes of the form $4n - 1$.
7. Let n be a positive integer. Suppose that 2^n and 5^n *begin* with the same digit. Prove that there is only one possible value for this common initial digit (and find it).
8. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.
9. Let f be a polynomial with positive integer coefficients and of positive degree. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.
10. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)