PUTNAM 2014 WEEK 9
REAL ANALYSIS

Some basic results from real analysis

Continuity. Remember these via pictures! The Intermediate Value Theorem (if f is continuous on $[a,b]$, then every value between $f(a)$ and $f(b)$ is of the form $f(c)$ for $a \leq c \leq b$). The Extreme Value Theorem (a continuous function on $[a,b]$ attains its sup and inf). Rolle's Theorem (if f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b)$, then there is a point $u \in (a,b)$ at which $f'(u) = 0$). The Mean Value Theorem (if f is continuous on $[a,b]$ and differentiable on (a,b), then there is a point $u \in (a,b)$ at which $f'(u) = \frac{f(b)-f(a)}{b-a}$.

Convergence. A bounded monotone sequence converges. A series in which the entries have alternating sign and which decrease in absolute value converge. A monotone sum whose corresponding integral is bounded converges (the integral comparison test). A sequence bounded above and below by two other convergent (to the same limit) sequences must converge (the squeeze principle).

Riemann Sums. The definite integral of a continuous function is the limit of the Riemann sums.

Easier Problems

1. Find a nice formula for the n-th derivative of the product $f(x)g(x)$.

2. Recall integration by parts:

$$\int f\, dg = fg - \int g\, df.$$

Substitute $f(x) = 1/x$, $g(x) = x$, and manipulate, to get

$$\int \frac{1}{x}\, dx = 1 + \int \frac{1}{x}\, dx.$$

Hence $0 = 1$. What has gone wrong?

3. Show that right now, there are two diametrically-opposed points on Earth that have exactly the same temperature

3. (a) Suppose $f(x)$ is a polynomial of odd degree. Then $f(x) = 0$ has a real root. (b) Any square matrix with an odd number of rows has a real eigenvalue.

4. Compute

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}.$$

5. Let $a, b, c, d, e \in \mathbb{R}$ such that

$$a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} = 0.$$

Show that the polynomial

$$a + bx + cx^2 + dx^3 + ex^4$$

has at least one real zero.

Harder Problems

6. Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

7. What happens if you put a random positive number in your calculator, and repeatedly hit the sequence of buttons "$1/x$", "+" and "1"? (In other words, what happens if you iterate $x \mapsto 1/x + 1$?)

8. Compute

$$\int_0^{\pi/2} \left[\sin^2(\sin x) + \cos^2(\cos x)\right]\, dx.$$

9. Prove that

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}}$$

10. Is $\sqrt{2}$ the limit of a sequence of numbers

$$a_1, a_2, a_3, \dots,$$

where each term a_i has a form $\sqrt[3]{n} - \sqrt[3]{m}$ for some non-negative integers n, m?