PUTNAM 2014 WEEK 8: NUMBER THEORY.

Basic number theory concepts to remember: modular arithmetic, unique factorization, greatest common divisor (a,b) can be written as a linear combination ax+by, Chinese Remainder Theorem, positional notation, Fermat's little theorem $a^p \equiv a \pmod{p}$.

Easier Problems

1. Find the smallest positive integer n such that one half of n is a square, one third of n is a cube, and one fifth of n is a fifth power.

2. Compute the last two digits of 47^{99}.

3. If $2n + 1$ and $3n + 1$ are both perfect squares, show that $40|n$.

4. Let n be a positive integer. Suppose that 2^n and 5^n *begin* with the same digit. Then there is only one possible value for this common initial digit. Find, with proof, that digit.

5. Let p_n denote the nth prime, and let π_n the count of primes less than n. For example:

$$\begin{aligned} \{p\}: &\quad 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots \\ \{\pi\}: &\quad 0, 0, 1, 2, 2, 3, 3, 4, 4, 4, \ldots \end{aligned}$$

Let q_n denote the number of terms of π less than n. What can you say about q_n? (Try a few small cases!) Why is this true?

Harder Problems

6. Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

7. Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.)

8. Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

9. Let p be a prime number. Show that

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}.$$

10. Prove that there are exactly three right-angled triangles T whose sides are integers and such that the area of T is equal to the perimeter of T (the triangle with sides 6, 8, 10 is one example).