PUTNAM 2014 WEEK 7
LINEAR ALGEBRA (MATH 235)

Syllabus of a good first linear algebra course

Systems of linear equations (Gauss' elimination). Matrices. Rank. Determinants. Inverse matrix. Adjoint matrix and Kramer's rule. Vector spaces. Linear span. Linear independence. Basis. Dimension. Linear transformations. Characteristic and minimal polynomials. Eigenvalues and eigenvectors. Trace. Jordan normal form. Orthogonal basis (Gram–Shmidt algorithm). Orthogonal transformations.

Easier Problems

1. It is well-known that all real-valued functions on $\mathbb{R}$ form a vector space. Does the function $\sin x$ belong to the linear span of the functions $1, \cos x, \cos 2x, \cos 3x, \ldots$? How about the function $\sin^2 x$?
2. $1/(x+1)(x+2)(x+3) = ?/(x+1) + ?/(x+2) + ?/(x+3)$.
3. Show that

$$\det \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \neq 0$$

if a, b, c are different numbers.
4. Let's call a real 3×3 matrix a "magic square" if all its row-sums and column-sums are equal to 0. Show that all magic squares form a vector space. Find its dimension.
5. Find all solutions of the system of infinitely many linear equations

$$\left(1 - \frac{1}{2^k}\right) x + \left(1 - \frac{1}{2^{k+1}}\right) y + \left(1 - \frac{1}{2^{k+2}}\right) z = 0$$

(one equation for each $k = 0, 1, 2, \ldots$).
6. Compute the determinant of the $n \times n$ matrix $[a_{ij}]$ such that

$$a_{ij} = |i - j|.$$

Harder Problems

7. Let A be a 3×3 matrix with integral coefficients such that all 2×2 minors of A are divisible by 5. Show that $\det A$ is divisible by 25.

8. The dashboard of a nuclear power station has several lights. Some lights are on and some are off. There are also several buttons. Pressing each button changes the state of several lights (from on to off and from off to on). It is known that for every set of lights there exists a button connected to an odd number of lights in this set. Show that one can turn off all lights by pressing some buttons.

9. Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of $\cos$ is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

10. Let A be a square $n \times n$ matrix such that

$$|a_{ii}| > |a_{i1}| + \ldots + |a_{i,i-1}| + |a_{i,i+1}| + \ldots + |a_{i,n}|$$

for every i. Show that $\det A \neq 0$.

11. For what λ does the integral equation (with unknown function f)

$$\int_0^1 \min(x,y) f(y)\, dy = \lambda f(x)$$

have continuous solutions which do not vanish identically on $[0,1]$?

12. Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of the unit hypercube in $\mathbb{R}^n$.) Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.