PUTNAM 2014 WEEK 5: PROBABILITY.

Easier Problems

1. I've just heard a public announcement that 80% of UMass students don't drink (to oblivion), 70% don't get high (on a regular basis), and 51% don't party (every night). Prove mathematically that at least one UMass student came here to study.

2. What's larger, the percentage of overfilled PVTA buses or the percentage of passengers that have to ride overfilled PVTA buses?

3. King Arthur is sick of fights for inheritance and decides to announce the following law. From now on, no family will be allowed to have another child after a boy is born. What will happen to the percentage of males in the general population in the long run?

4. What are the chances that a five-card poker hand is a flush? a straight?

5. The temperatures in Amherst and Boston are x° and y°, respectively. We are given

- $P(x^\circ = 70^\circ)$, the probability that the temperature in Amherst is 70°,
- $P(y^\circ = 70^\circ)$, and
- $P(\max(x^\circ, y^\circ) = 70^\circ)$.

Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$.

Harder Problems

6. You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of landing on heads. If the n coins are tossed, what is the probability that the number of heads is odd?

7. Three real numbers a, b, c are randomly (and uniformly) chosen from the interval $[0, 1]$. What is the probability that there exists a triangle with sides a, b, c?

8. Several chords are constructed in a circle of radius 1. Prove that if every diameter intersects at most k chords then the total length of chords is $\leq k\pi$.

9. Let p_n be the probability that $c + d$ is a perfect square, where the integers c and d are selected independently at random from the set $\{1, \ldots, n\}$. Find the limit

$$\lim_{n\to\infty} p_n \sqrt{n}.$$

10. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.