## PUTNAM 2014 WEEK 4: GRAPH THEORY.

A graph is a collection of vertices and a collection of edges connecting some vertices. We can visualize a graph by drawing a picture where vertices are dots and edges are paths connecting the dots. Just about any situation involving "relationships" between "objects" can be recast as a graph, where the vertices are the "objects" and we join vertices with edges if the corresponding objects are "related".

**Easier Problems**

**1.** Some people in a room are friends with each other and others are not. Show that the number of people having an odd number of friends is even.

**Terminology.** A graph contains a *cycle* if we can find some vertices $v_1, \ldots, v_k$ such that $v_1$ is connected to $v_2$, $v_2$ is connected to $v_3$, ..., $v_{k-1}$ is connected to $v_k$, and $v_k$ is connected to $v_1$. A graph without cycles is called a *tree*.

**2.** Show that a tree with $n$ vertices contains at most $n - 1$ edges. Is the converse true? Is the estimate sharp?

**3.** Is it possible to draw the figure on the right without lifting your pencil in such a way that you never draw the same line twice?

**Terminology.** A graph is called *Eulerian* if there exists a path that traverses every edge exactly once. For the historical perspective, see `http://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg`

**4.** Is it possible for a knight to travel around a standard $8 \times 8$ chessboard, starting and ending in the same square, in such a way that the knight completes every possible move exactly once? We consider a move between two squares to be completed if it occurs in either direction.

**5.** Given six people, show that among them there is a set of three mutual friends, or a set of three complete strangers.

**6.** In a group of nine people, one person knows exactly two others, two people know exactly 5 others, and another 2 people know exactly 6 others. Show that there are three people who all know each other.

**Harder Problems**

**7.** At a party, assume that no boy dances with every girl but each girl dances with at least one boy. Prove that there are two couples $gb$ and $g'b'$ which dance together such that $b$ does not dance with $g'$ nor does $g$ dance with $b'$.

**8.** Consider a set of $2n$ points in space. Suppose they are joined by at least $n^2 + 1$ segments. Show that at least one triangle is formed.

**9.** Some people in a room are friends with each other and others are not. Show that it is possible to put hats on some people so that people wearing hats are friends with at least as many people not wearing hats as with people wearing hats, and people not wearing hats are friends with at least as many people wearing hats as with people not wearing hats.

**10.** During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that at some point, at least three of them were sleeping simultaneously.

**11.** Let $n$ be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the $k$-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

**12.** Prove that a list can be made of all the subsets of a finite set in such a way that (i) the empty set is first on the list, (ii) each subset occurs exactly once, (iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

**Terminology.** The graph is called *Hamiltonian* if there exists a path that passes through every vertex exactly once. The last problem can be rephrased as a statement that a certain graph is Hamiltonian. Which graph is it?