# PUTNAM 2014 WEEK 3: GEOMETRY, VECTORS, COMPLEX NUMBERS.

It is unfortunate that American kids learn less high school geometry than their peers in other countries – because geometry is a lot of fun! The textbook has a great section "Geometry for Americans", which is very helpful. There are three basic approaches to geometric problems:

(1) Axiomatic approach, where everything is deduced from basic facts, such as congruence tests for triangles (SAS, ASA, and SSS), similarity of triangles, angles in the circle theorem, etc. These problems are great to sharpen your proof skills but you are unlikely to see them on the Putnam exam.
(2) Method of coordinates. Points on the plane are interpreted as coordinates $(x, y) \in \mathbb{R}^2$, or vectors, or complex numbers. Calculations can often be simplified by using basic linear algebra (scalar products, etc.) and knowing geometric interpretations of various algebraic operations (e.g. multiplication of complex numbers). Alternatively, a lot of things can be computed using trig functions.
(3) Symmetries and transformations. This is a more dynamic approach, where you apply and compose rotations, symmetries, etc.

Very often problems are only formulated using geometric language, but the proof uses some counting trick, or combinatorics, etc.

**Easier Problems**

**1.** Let $ABC$ be a triangle and let the angle bisector of $\angle A$ intersect the side $BC$ at a point $D$. Show that $AB/AC = BD/CD$.

**2.** Find (with proof!) the length of the shortest path from the point $(3, 5)$ to the point $(8, 2)$ that touches the $x$-axis and also touches the $y$-axis.

**3.** Given an arbitrary triangle $ABC$, construct equilateral triangles $ABZ$, $ACY$ and $BCX$ externally on the sides of the triangle $ABC$. Prove that $AX = BY = CZ$.

**4.** Prove that two lines AB and CD are perpendicular if and only if

$$AC^2 - AD^2 = BC^2 - BD^2.$$

**5.** Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?

**6.** Points $A$ and $B$ move clock-wise around circumferences $C_1$ and $C_2$ with equal angular speeds. Prove that the vertex $C$ of the equilateral triangle $ABC$ also moves around some circumference.

**Harder Problems**

**7.** Right triangle $ABC$ has a right angle at $C$ and $\angle BAC = \theta$; the point $D$ is chosen on $AB$ so that $AC = AD = 1$; the point $E$ is chosen on $BC$ so that $\angle CDE = \theta$. The perpendicular to $BC$ at $E$ meets $AB$ at $F$. Evaluate

$$\lim_{\theta \to 0} EF.$$

**8.** Show that there do not exist any equilateral triangle in the plane whose vertices are lattice points (i.e. have integer coordinates).

**9.** Given that $A$, $B$, and $C$ are noncollinear points in the plane with integer coordinates such that the distances $AB$, $AC$, and $BC$ are integers, what is the smallest possible value of $AB$?

**10.** Let $n$ be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the $xy$-plane, for $k = 1, \ldots, n$. Let $R_k$ be the map that rotates the plane counterclockwise by the angle $\theta$ about the point $P_k$. Let $R$ denote the map obtained by applying, in order $R_1$, then $R_2$, ..., then $R_n$. For an arbitrary point $(x, y)$, find, and simplify, the coordinates of $R(x, y)$.

**11.** Let $n$ be a positive integer. Find the number of pairs $P, Q$ of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and $\deg P > \deg Q$.

**12.** For $k = 1, \ldots, n$, let $z_k = x_k + iy_k$, where $x_k$ and $y_k$ are real. Let $r$ be the absolute value of the real part of

$$\pm\sqrt{z_1^2 + \ldots + z_n^2}.$$

Prove that $r \leq |x_1| + \ldots + |x_n|$.