# PUTNAM 2014 WEEK 2. GAMES AND INVARIANTS.

## Easier Problems

**1.** Beavis and Butthead play "double chess": all the rules are like in usual chess but every player makes two moves every time. Prove that the white side is guaranteed at least a draw.

**2.** Alice and Bob play a game where at each step one of them puts a stone in one of the squares of the $8 \times 8$ board. A player loses if he creates an $L$-shape of three stones. What is a winning strategy?

**3.** For his birthday, JT gets a square cake with a candle in the middle. He allows his first guest to cut a triangular piece by cutting two adjacent sides of the square away from the vertices. The next guest can cut a triangular piece from what's left. But again, he can only cut two consecutive sides away from the vertices, and so on. Will any of the guests be able to get a piece with a candle?

**4.** Two players play the following game. They start with two piles of candy, one player with 21 and the other with 20 candies. At each turn, a player eats one of the piles and splits the other into two piles. The first player who can't make a legal move loses. Who will win, the first player or the second player?

**5.** Let $P_1, \ldots, P_{1001}$ be distinct points in the plane. Connect the points with the line segments $P_1P_2$, $P_2P_3$, $\ldots$, $P_{1000}P_{1001}$, $P_{1001}P_1$. Is it ever possible to draw a line that passes through the interior of every one of these segments?

**6.** Let $S$ be a set of points in the plane and let $R$ be a fixed positive real number. We want to place discs of radius $R$ on the plane such that

(i) Every point in $S$ is covered by exactly one disc, and
(ii) Every disc is centered at a point in $S$.

This may or may not be possible, but if it is possible, show that all solutions use the exact same number of discs.

**Harder Problems**

**7.** A wolf and a hundred sheep play the following game. A wolf jumps 1 yard (or less) in any direction. Then one (and only one!) sheep jumps 1 yard in any direction, and so on. Is it true that the wolf can always catch one of the sheep?

**8.** A game is played as follows. The first player selects an interval $[a, b]$. The second player selects an interval $[c, d] \subset [a, b]$. The first player selects an interval inside $[c, d]$, and so on. The game goes on forever. The first player will win if the intersection of all segments contains a rational number. Is he going to win?

**9.** Suppose $2007$ red points and $2007$ blue points are given in the plane, no three collinear (and pairwise distinct). Show that one can match up the red points and the blue points with line segments so that no two line segments intersect.

**10.** Start with three stones located at coordinates $(0, 0)$, $(1, 0)$, and $(0, 1)$. At each step, you must increase the number of stones in the following way. Remove one of the stones (with coordinates $(x, y)$) and instead put two stones, one at $(x + 1, y)$, and one at $(x, y + 1)$, assuming there are no stones at these coordinates. Is it possible to devise a sequence of steps so that in the end all three initially occupied coordinates will be empty?

**11.** A game of solitaire is played as follows. After each play, according to the outcome, the player receives either $a$ or $b$ points ($a$ and $b$ are positive integers with $a > b$) and his score accumulates from play to play. It has been noticed that there are thirty-five unattainable scores and that one of these is $58$. Find $a$ and $b$.

**12.** An elephant and a donkey play the following game. They start with 2012 points on the plane such that no three lie on a line. Every two points are connected by a segment. The elephant can mark any of the points with one of the digits $0, \ldots, 9$. The donkey can mark any of the segments with one of the digits. Of course at some point the elephant will be out of moves, at which point the donkey will just continue marking segments. The donkey will win if in the end one of the segments will be marked by the same digit as both of its endpoints. Who is going to win?