PUTNAM 2014 WEEK 11: COMBINATORICS

Easier Problems

Binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ form the Pascal triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

where each number is equal to the sum of two numbers above it. Also, $\binom{n}{k}$ is equal to the number of ways to choose k objects out of n.

1. Consider a triangle

$$\begin{array}{ccccccccc} & & & & \frac{1}{1} & & & & \\ & & & \frac{1}{2} & & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & \\ & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & \\ \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} \end{array}$$

where each number is equal to the sum of two numbers *below* it. Find out how numbers in this triangle are related to binomial coefficients.

2. Find lines in the Pascal triangle that add up to Fibonacci numbers.

Some problems in this set can be solved using the method of bijections: *Two sets have the same number of elements if there is a bijection between them.* For example, $\binom{n}{k} = \binom{n}{n-k}$ because the number of ways to choose k objects out of n is equal to the number of ways to choose a complementary subset of $n-k$ objects.

3. (a) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ using method of bijections.
(b) Show that $\binom{n+m}{k} = \binom{n}{k}\binom{m}{0} + \binom{n}{k-1}\binom{m}{1} + \binom{n}{k-2}\binom{m}{2} + \ldots + \binom{n}{0}\binom{m}{k}$.

4. Show that $\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$.

5. Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

6. Find all lines in the Pascal triangle which contain only odd integers (for example the 4-th line $\quad 1 \quad 3 \quad 3 \quad 1 \quad$).

Harder Problems

7. (a) Any infinite sequence of real numbers contains either an infinite increasing subsequence or an infinite decreasing subsequence.
(b) Show that any sequence of real numbers of length $(n-1)^2+1$ contains a monotonically increasing subsequence or a monotonically decreasing subsequence of length n.
8. Prove that $\binom{2k}{k} = \frac{2}{\pi}\int_0^{\pi/2}(2\sin x)^{2k}\,dx$.

The remaining problems in this worksheet can be solved using generating functions. Given a sequence $a_0, a_1, a_2, \ldots$, the power series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \ldots$$

is called a generating function of the sequence. For example, $f(x) = (1+x)^n$ is the generating function of the sequence

$$\binom{n}{0},\ \binom{n}{1},\ \binom{n}{2},\ \binom{n}{3},\ \ldots$$

because, by the binomial fomula,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^n + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n.$$

9. Solve Problem 3(b) using the identity $(1+x)^n(1+x)^m = (1+x)^{n+m}$. (Hint: compute the x^k term on both sides of the identity).
10. Give a closed-form formula for the sum $\sum\limits_{i=1}^{n} i^2\binom{n}{i}$.
11. (a) Prove that

$$f(x) = \frac{1}{1-x-x^2}$$

is the generating function for the Fibonacci sequence $1, 1, 2, 3, 5, \ldots$.
(b) Write $f(x)$ using partial fractions and deduce from it a closed-form formula for the Fibonacci sequence.
12. Show that for each positive integer n, the number of partitions of n into unequal parts is equal to the number of partitions of n into odd parts. For example, $n=6$ has 4 partitions into unequal parts:

$$1+5, \quad 1+2+3, \quad 2+4, \quad 6.$$

And there are also 4 partitions into odd parts:

$$1+1+1+1+1+1, \quad 1+1+1+3, \quad 1+5, \quad 3+3.$$